Electromagnetics

Reference	Differential Form	Integral Form
Gauss's law	$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$
Faraday's law	$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \dfrac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$
No magnetic charges (Gauss's law for magnetism)	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$
Ampère's law	$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{s}$





Statics

stat·ic
adj.
1. a. Having no motion; being at rest; quiescent.
b. Fixed; stationary.
2. *Physics* Of or relating to bodies at rest or forces that balance each other.
3. *Electricity* Of, relating to, or producing stationary charges; electrostatic.
4. Of, relating to, or produced by random radio noise.
n.
1. Random noise, such as crackling in a receiver or specks on a television screen, produced by atmospheric disturbance of the signal.
2. *Informal*
a. Back talk.
b. Interference; obstruction.
c. Angry or heated criticism.

$$\nabla \cdot \mathbf{D} = \rho_v$$

$$\nabla \times \mathbf{E} = -\cancel{\frac{\partial \mathbf{B}}{\partial t}} = 0$$

ElectroStatics

$$\nabla \cdot \mathbf{B} = 0$$

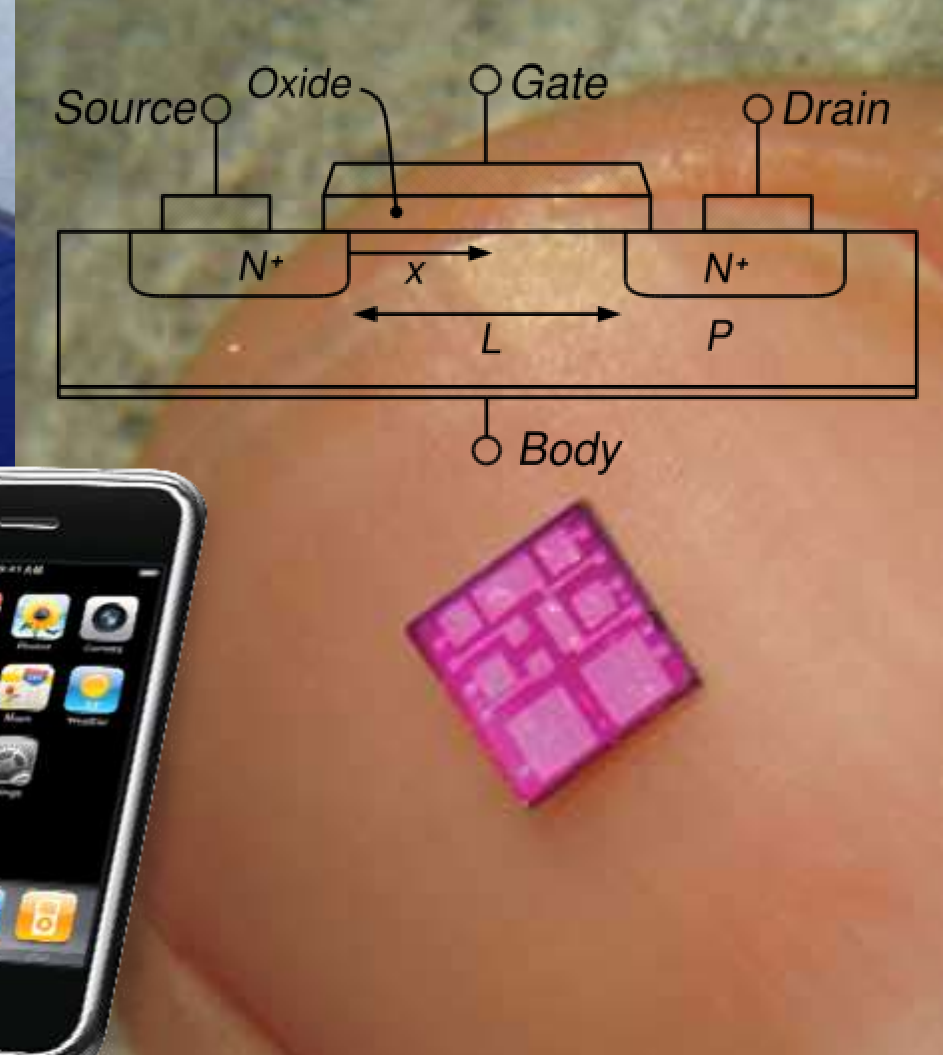
$$\nabla \times \mathbf{H} = \mathbf{J} + \cancel{\frac{\partial \mathbf{D}}{\partial t}} = \vec{J}$$

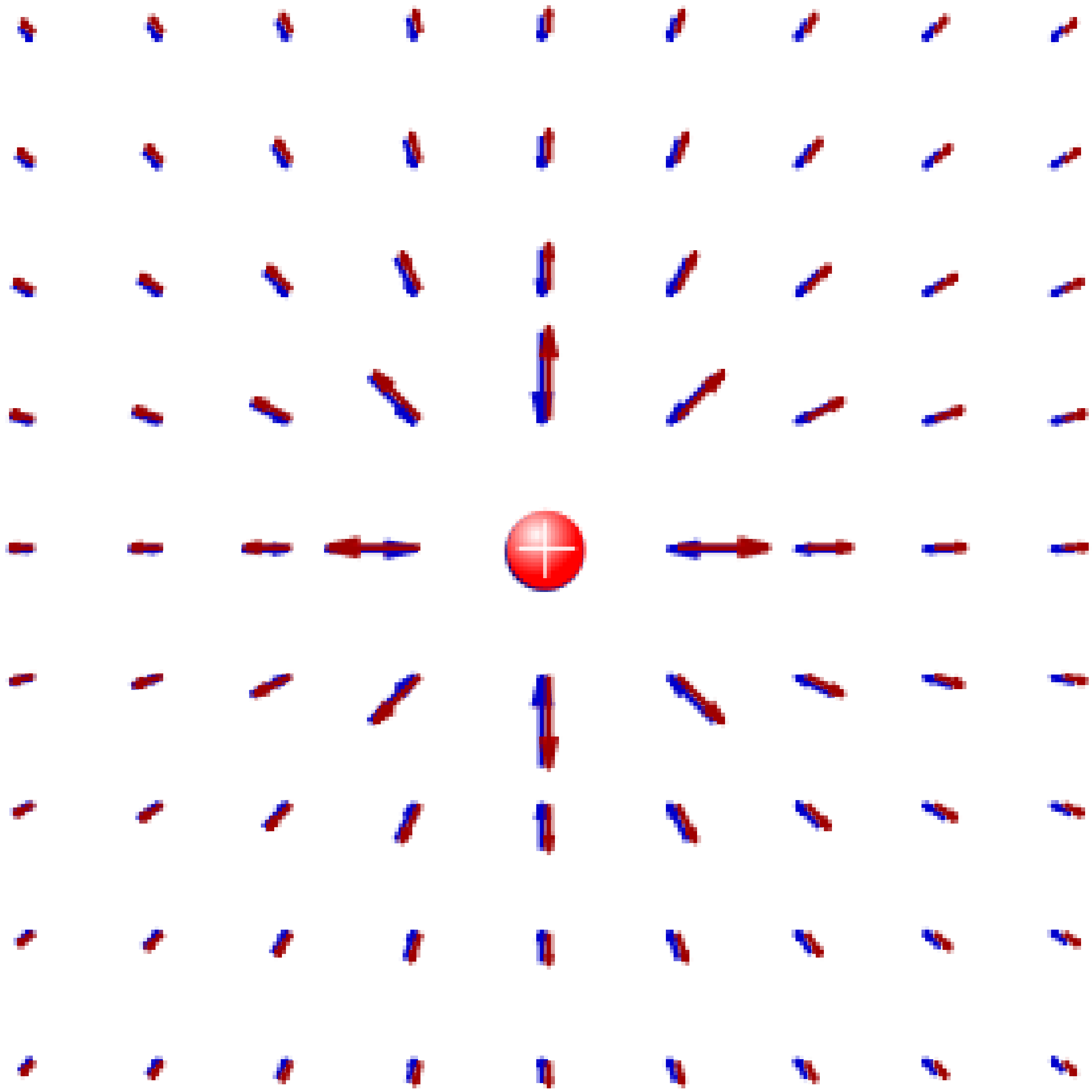
MagnetoStatics

For the special case of **no** time variations (i.e. *statics*) the electric and magnetic fields are de-coupled, and we can treat them separately!

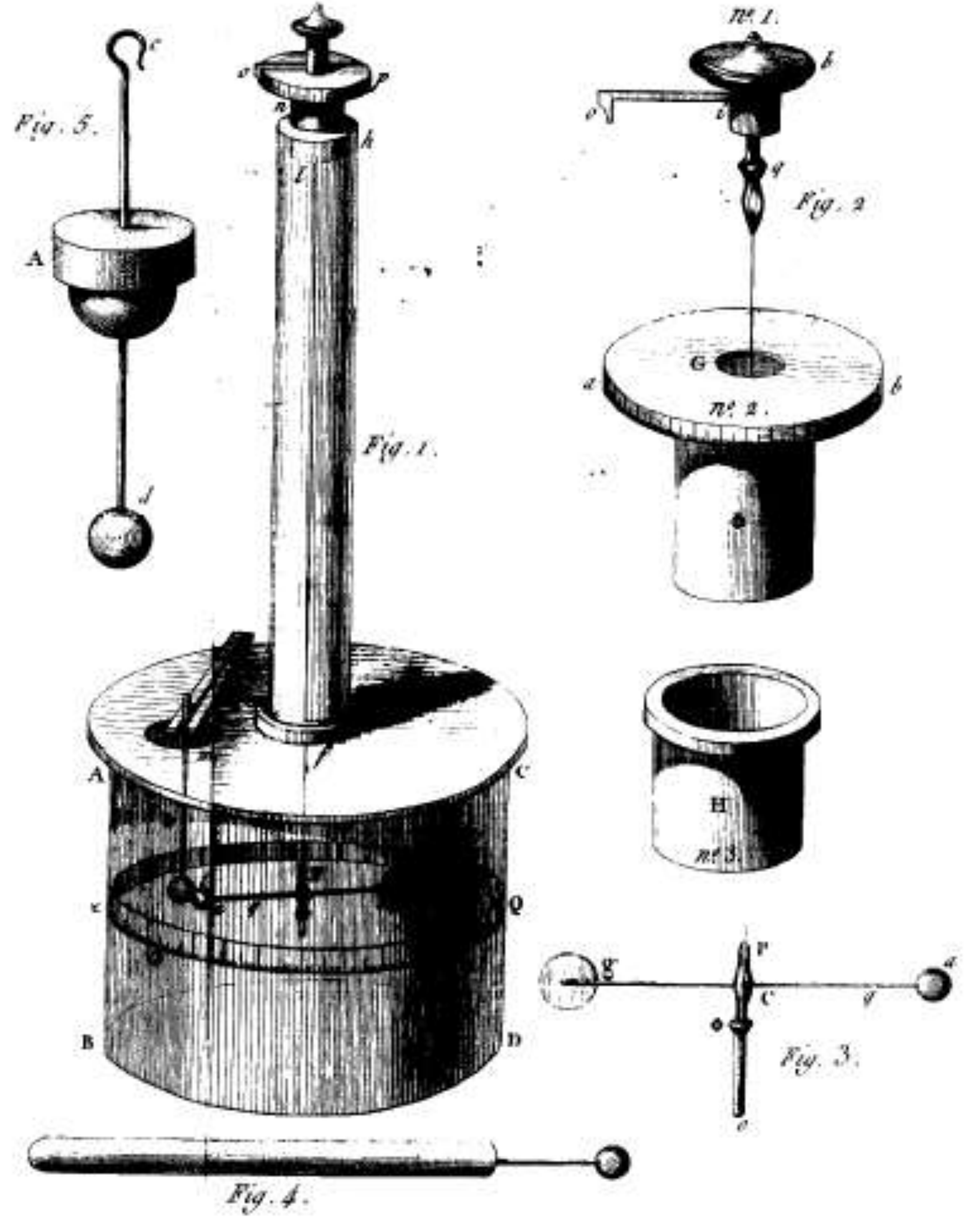
6 Feb.	**Electrostatics**	4-1 thru 4-3
8 Feb.	"	4-4
10 Feb.	"	4-5
13 Feb.	"	4-6 thru 4-8 **Lab: Cap Design**
15 Feb.	"	4-9, 4-10
17 Feb.	"	4-11 *(and review)*

Electrostatics





Coulomb's Law

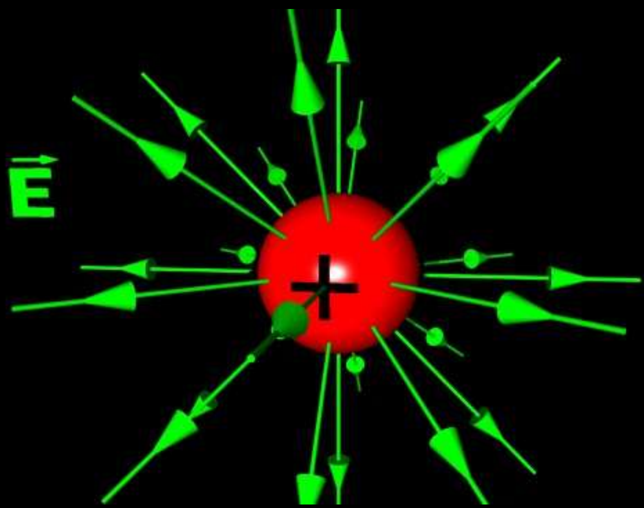


vector field

$$\vec{E} = \hat{R}\,\frac{q}{4\pi\varepsilon R^2}$$

$$\vec{F} = q'\vec{E}$$

different q



$$\vec{E} = \hat{R}\,\frac{q}{4\pi\varepsilon R^2}$$

$$\nabla \cdot \mathbf{D} = \rho_v$$

$$\nabla \times \mathbf{E} = 0$$

Statics only ☺

$$\vec{D} = \epsilon\,\vec{E}$$

Electric Flux Density

Permittivity

Electric Field Intensity

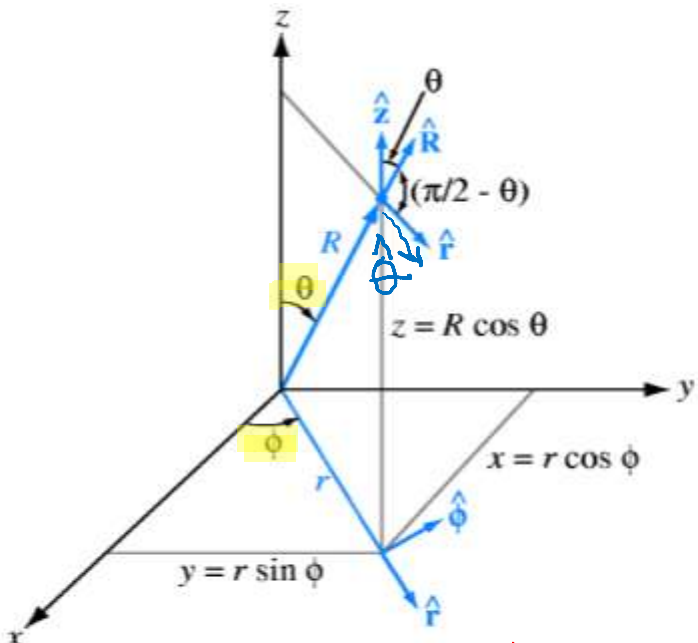
A function of material...

... in free-space (i.e. Vacuum), $\epsilon_0 \approx 8.85\times10^{-12}\ \frac{F}{m}$

Vector field **E** is given by

$$\mathbf{E} = \hat{\mathbf{R}}\, 5R\cos\theta - \hat{\boldsymbol{\theta}}\frac{12}{R}\sin\theta\cos\phi + \hat{\boldsymbol{\phi}}3\sin\phi.$$

Determine the component of **E** tangential to the spherical surface $R = 2$ at point $P(2, 30°, 60°)$.



Solution: At P, **E** is given by

$$\mathbf{E} = \hat{\mathbf{R}}\, 5\times 2\cos 30° - \hat{\boldsymbol{\theta}}\frac{12}{2}\sin 30°\cos 60° + \hat{\boldsymbol{\phi}}3\sin 60°$$
$$= \hat{\mathbf{R}}\, 8.67 - \hat{\boldsymbol{\theta}}1.5 + \hat{\boldsymbol{\phi}}2.6.$$

The $\hat{\mathbf{R}}$ component is normal to the spherical surface while the other two are tangential. Hence,

$$\mathbf{E}_t = -\hat{\boldsymbol{\theta}}1.5 + \hat{\boldsymbol{\phi}}2.6.$$

Note: Some other texts define spherical coordinates a bit differently... be careful!

Charge Densities

are a function of location...

* Actually, let's shy clear of Quantum Scales for now ☺

$e = -1.602\text{x}10^{-19}$ *Coulombs*

$$\rho_v = \lim_{\Delta v \to 0^*} \frac{\Delta q}{\Delta v} = \frac{dq}{dv} \quad C/m^3$$

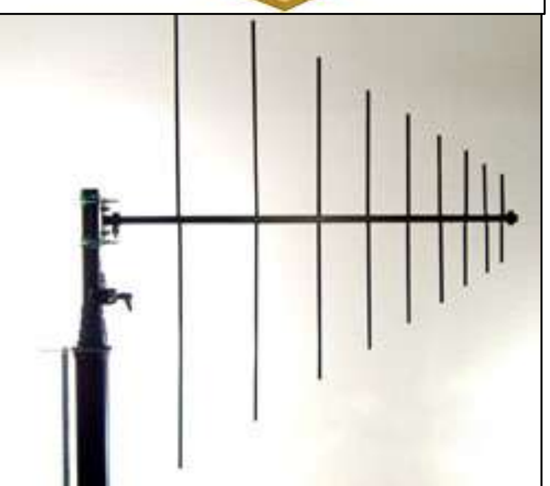
Volume

$$Q = \text{total charge} = \int_V \rho_v \, dv$$

$$\rho_s = \frac{dq}{ds} \quad C/m^2$$

Surface

$$Q = \int_S \rho_s \, ds$$



$$\rho_\ell = \frac{dq}{d\ell} \quad C/m$$

Line

$$Q = \int_L \rho_\ell \, d\ell$$

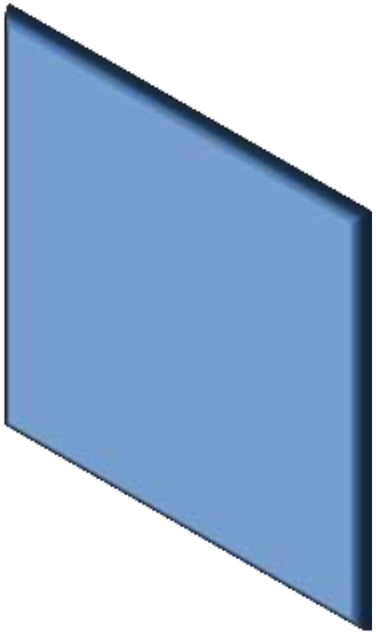
Exercise 4.1 A square plate in the x–y plane is situated in the space defined by $-3\text{ m} \le x \le 3\text{ m}$ and $-3\text{ m} \le y \le 3\text{ m}$. Find the total charge on the plate if the surface charge density is given by $\rho_s = 4y^2$ (μC/m^2).

$$\rho_s = 4y^2$$

$$Q = \int_S \rho_s \, ds$$

$$= \int_{-3}^{3} \int_{-3}^{3} 4y^2 \, dx \, dy$$

$$= \left. \left. \frac{4y^3 x}{3} \right|_{-3}^{3} \right|_{-3}^{3} = 432\ \mu\text{C} = 0.432 \quad (\text{mC}).$$



Exercise 4.2 A spherical shell centered at the origin extends between $R = 2$ cm and $R = 3$ cm. If the volume charge density is given by $\rho_v = 3R \times 10^{-4}$ (C/m^3), find the total charge contained in the shell.

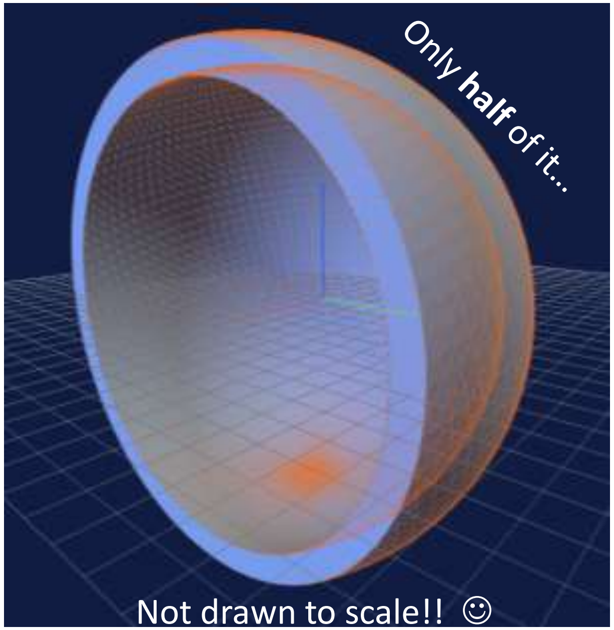
$$\rho_v = 3R \times 10^{-4}$$

$$Q = \int_{\mathcal{V}} \rho_v \, d\mathcal{V}$$

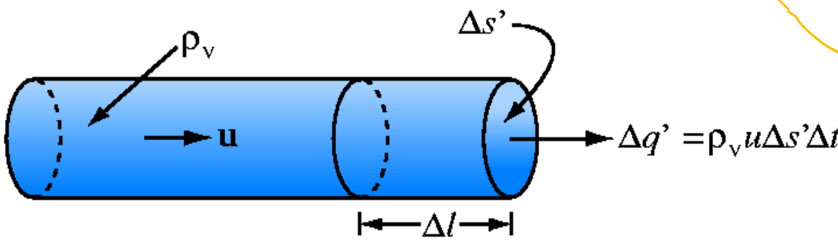
$$= \int_{R=2\text{ cm}}^{3\text{ cm}} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} 3R \times 10^{-4} \cdot R^2 \sin\theta \, dR \, d\theta \, d\phi$$

$$= \left. \frac{3R^4}{4} \times 10^{-4} \right|_{2\text{ cm}}^{3\text{ cm}} \times 2 \times 2\pi$$

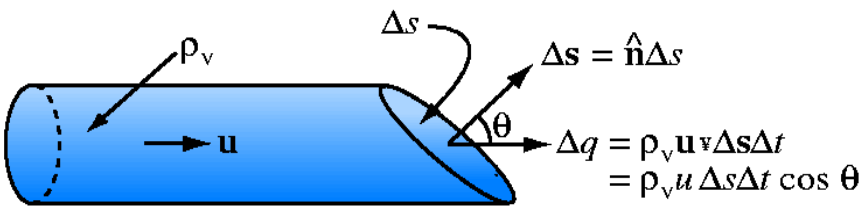
$$= 3\pi \times 10^{-4} \left[(3 \times 10^{-2})^4 - (2 \times 10^{-2})^4 \right] = 0.61 \quad (\text{nC}).$$



Current Density



(a)



(b)

C/m^3 m/s C/s

$$\vec{J} = \rho_v \vec{u} \quad A/m^2$$

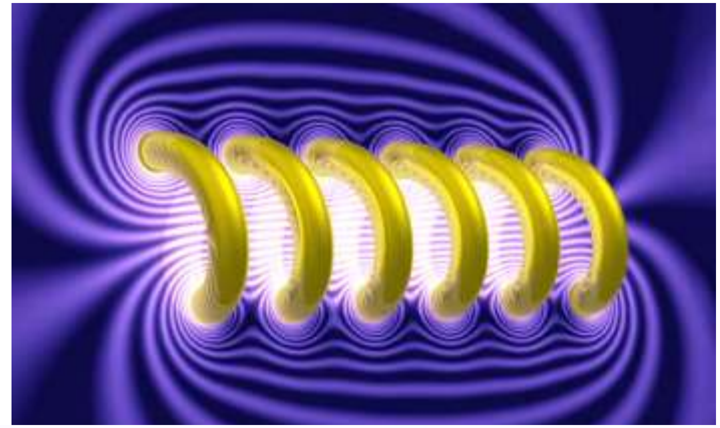
$$I = \int_S \vec{J} \cdot d\vec{s} \quad A$$

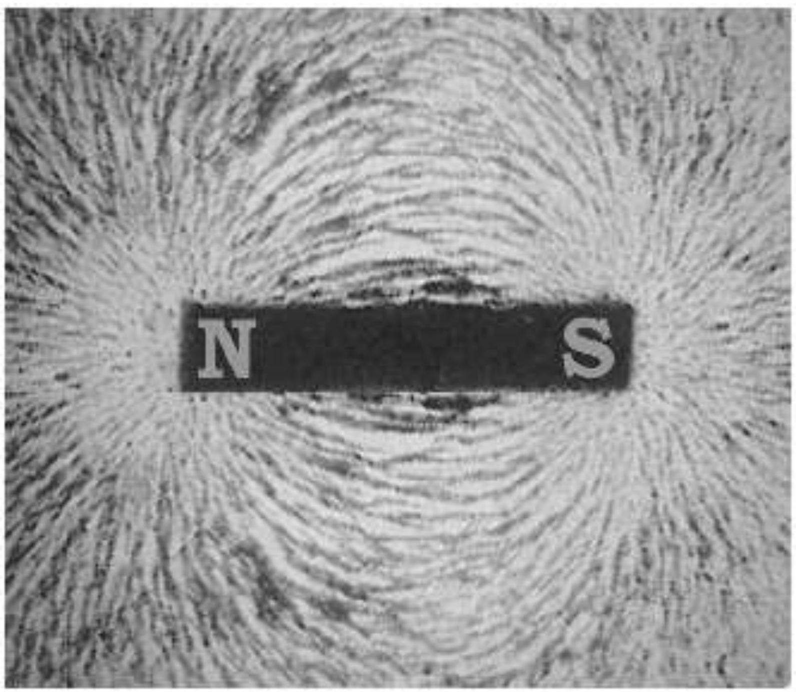
Current

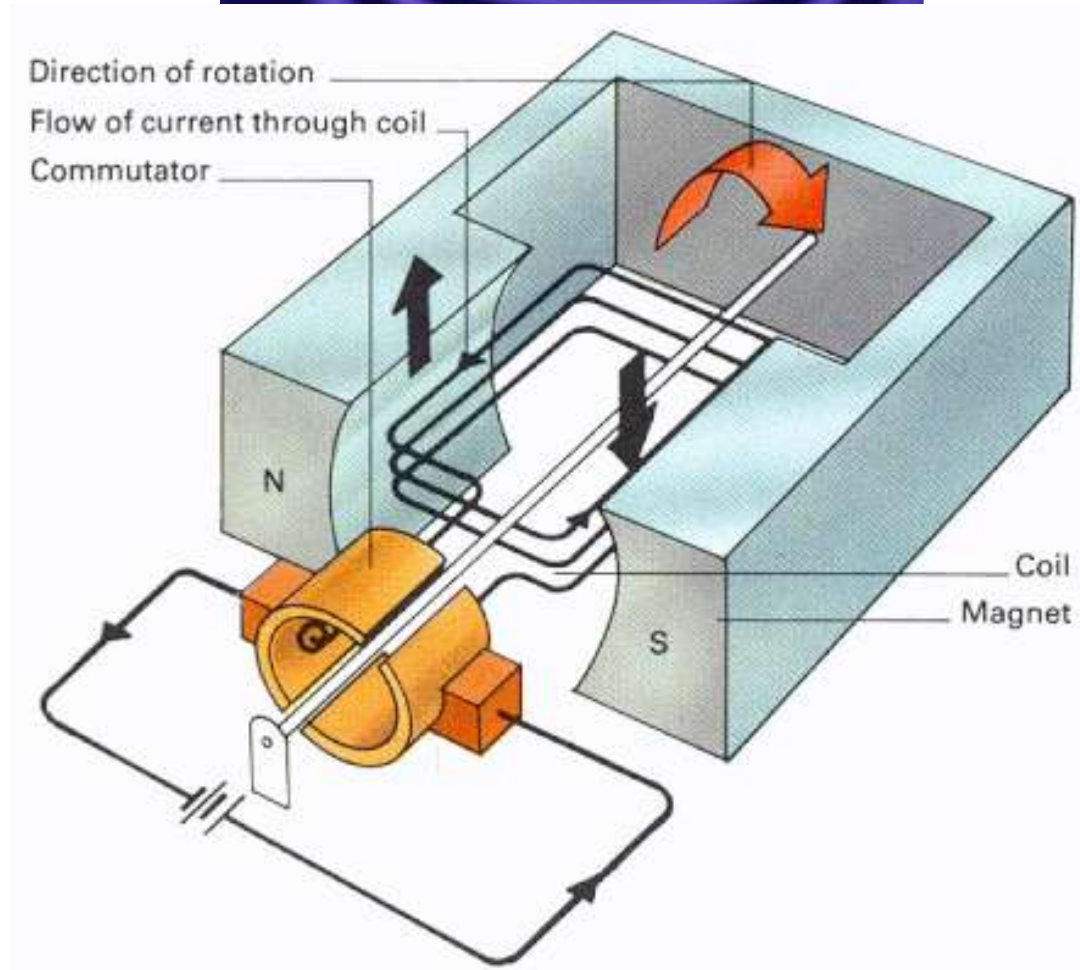
CONVECTION vs. conduction

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$







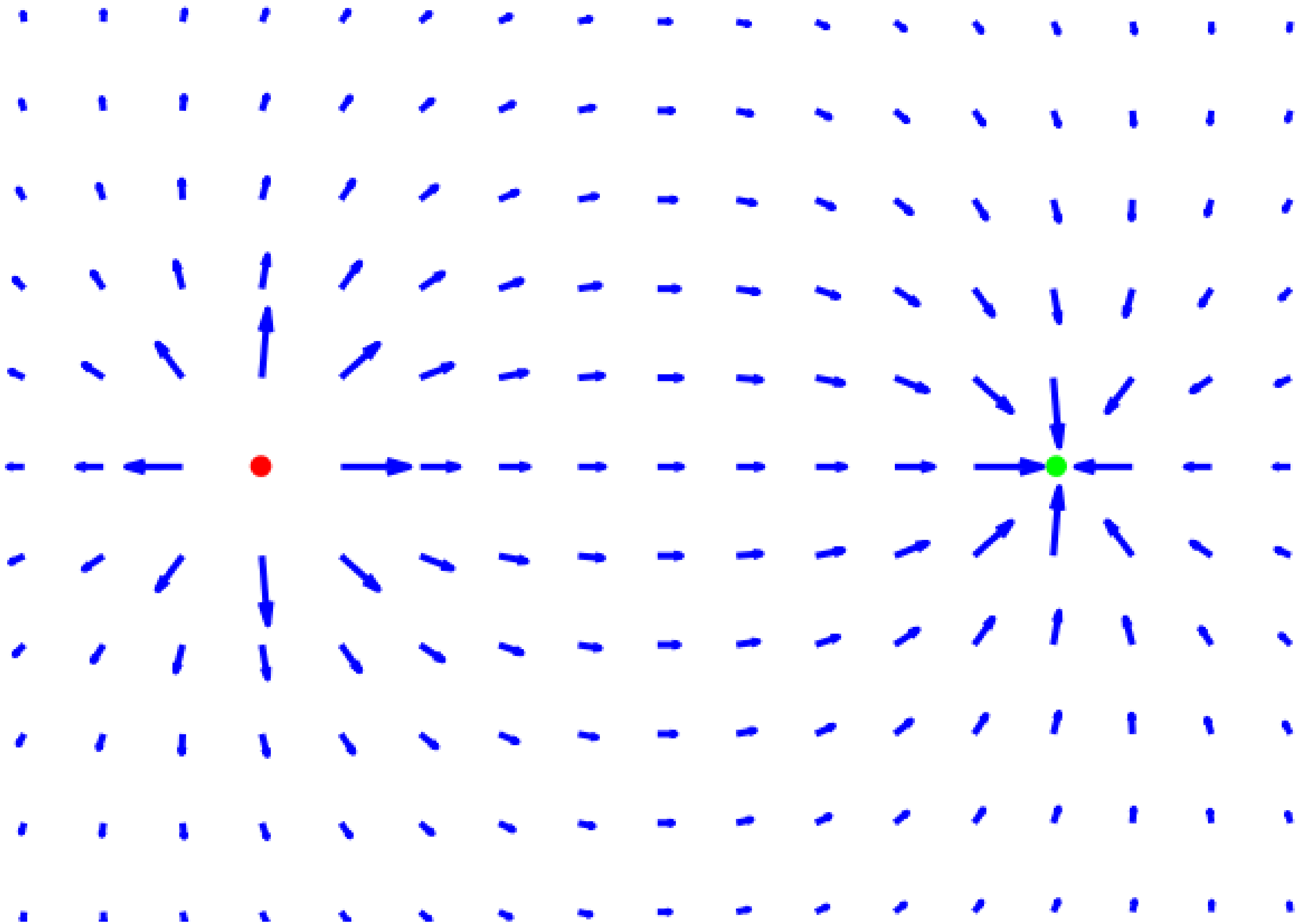
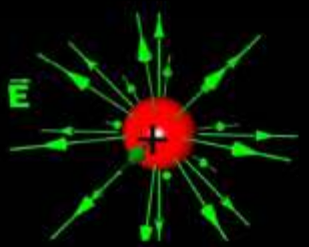


Illustration of the electric field surrounding a positive (red) and a negative (green) charge.

http://www.cco.caltech.edu/~phys1/java/phys1/EField/EField.html ← Note: field **lines**…

Point Charge



$$\vec{E} = \hat{R}\,\frac{q}{4\pi\epsilon R^2}$$

$$\vec{D} = \epsilon \vec{E}$$

Electric Flux Density

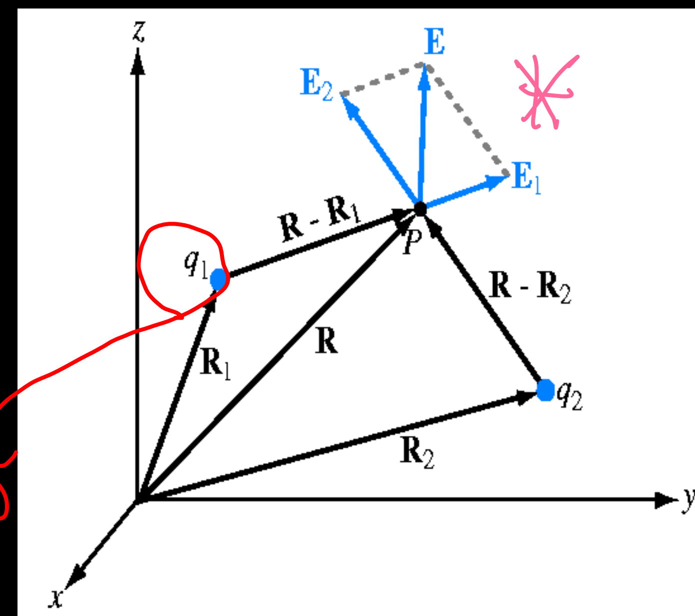
Permittivity

Electric Field Intensity

A function of material...

... in Free-space (i.e. Vacuum), $\epsilon_0 \approx 8.85\times10^{-12}\ \frac{F}{m}$

Multiple Point Charges



due to this charge

$$\vec{E}_1 = \frac{q_1(\vec{R}-\vec{R}_1)}{4\pi\epsilon\,|\vec{R}-\vec{R}_1|^3}$$

general case, point charge not at origin

SuperPosition

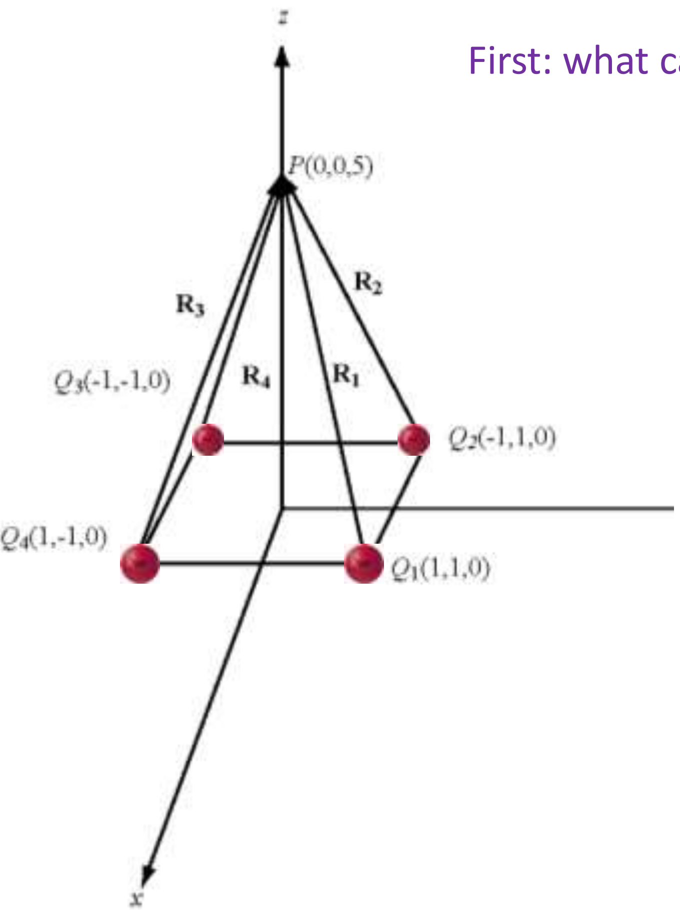
$\vec{E}_{TOTAL}$ = sum of all $\vec{E}$ contributions (if Linear media)

*Vector Addition

$$\Rightarrow \quad \vec{E}_{TOTAL} = \frac{1}{4\pi\epsilon}\sum_{i=1}^{N}\frac{q_i(\vec{R}-\vec{R}_i)}{|\vec{R}-\vec{R}_i|^3} \quad \frac{V}{m}$$

A square with sides 2 m each has a charge of 40 μC at each of its four corners. Determine the electric field at a point 5 m above the center of the square.

First: what can you tell be *qualitatively* about the electric field at that point??



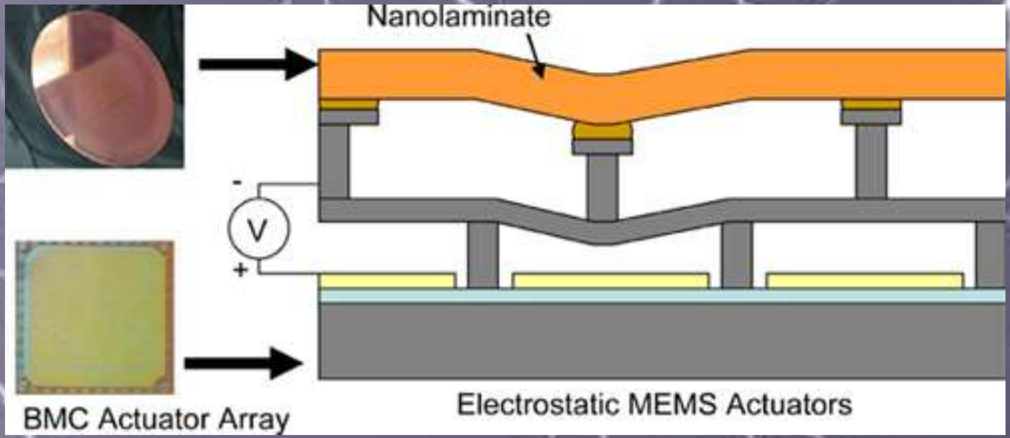
The distance $|R|$ between any of the charges and point P is

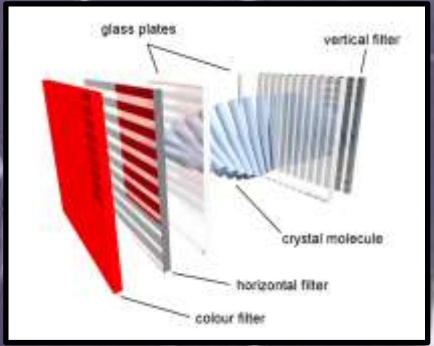
$$|R| = \sqrt{1^2+1^2+5^2} = \sqrt{27}.$$

$$\begin{aligned}\mathbf{E} &= \frac{Q}{4\pi\varepsilon_0}\left[\frac{\mathbf{R}_1}{|\mathbf{R}|^3}+\frac{\mathbf{R}_2}{|\mathbf{R}|^3}+\frac{\mathbf{R}_3}{|\mathbf{R}|^3}+\frac{\mathbf{R}_4}{|\mathbf{R}|^3}\right]\\ &= \frac{Q}{4\pi\varepsilon_0}\left[\frac{-\hat{\mathbf{x}}-\hat{\mathbf{y}}+\hat{\mathbf{z}}5}{(27)^{3/2}}+\frac{\hat{\mathbf{x}}-\hat{\mathbf{y}}+\hat{\mathbf{z}}5}{(27)^{3/2}}+\frac{-\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}}5}{(27)^{3/2}}+\frac{\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}}5}{(27)^{3/2}}\right]\\ &= \hat{\mathbf{z}}\frac{5Q}{(27)^{3/2}\pi\varepsilon_0} = \hat{\mathbf{z}}\frac{5\times 40\ \mu\text{C}}{(27)^{3/2}\pi\varepsilon_0} = \frac{1.42}{\pi\varepsilon_0}\times 10^{-6}\ (\text{V/m}) = \hat{\mathbf{z}}\,51.2\ (\text{kV/m}).\end{aligned}$$

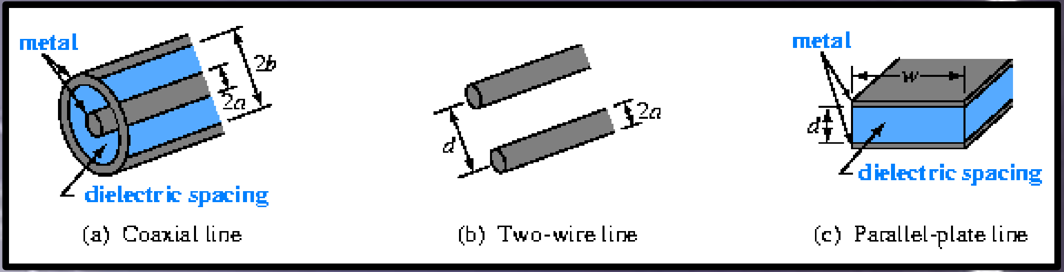
$$\nabla \cdot \mathbf{D} = \rho_v$$

$$\nabla \times \mathbf{E} = 0$$





$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

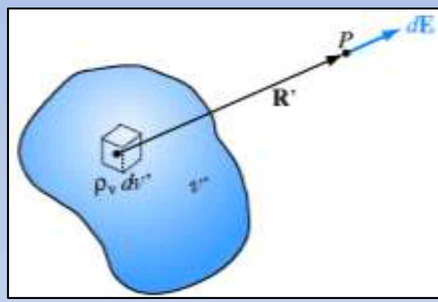


Electric Fields from Distributed Charge

Start with E-field due to *multiple* charges…

$$\vec{E}_{Total} = \frac{1}{4\pi\epsilon}\sum_{i=1}^{N \to \infty} \frac{q_i(\vec{R}-\vec{R}_i)}{|\vec{R}-\vec{R}_i|^3} \quad \frac{V}{m}$$

$\rho_v\,dv$ OR $\rho_s\,ds$ OR $\rho_\ell\,d\ell$ (depending on scenario)



…now, for brevity of notation (?): $\vec{R}' = R'\hat{R}'$ = vector from differential charge to point P

…so:

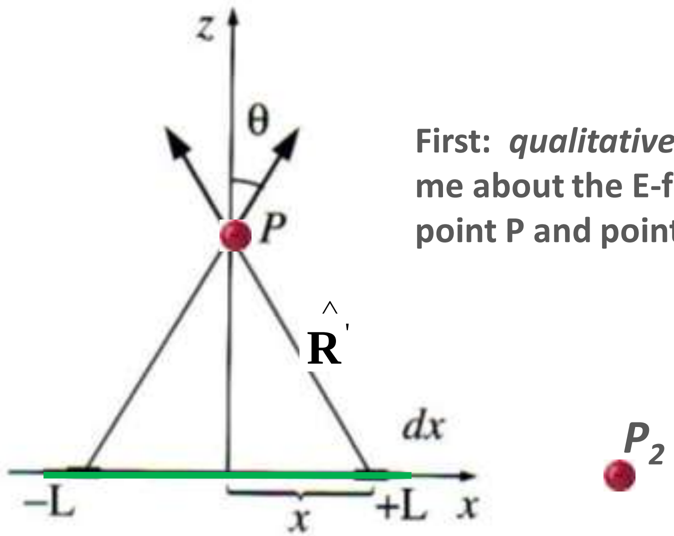
at point P

$$\vec{E} = \frac{1}{4\pi\epsilon}\int_V \hat{R}'\frac{\rho_v\,dv}{R'^2}$$

and analagous for surface & line charge distributions.

Example: Line charge…

Find the E-field a distance *z* above the midpoint of a straight line segment of length *2L* with a uniform line charge ρ_l.



First: *qualitatively* tell me about the E-field at point P and point P_2.

$$\vec{E} = \frac{1}{4\pi\epsilon}\int_{l} \hat{R}' \frac{\rho_l \, dl}{R'^2}$$

$$= \frac{\rho_l}{4\pi\epsilon}\int_{-L}^{L} \frac{\hat{R}'}{(x^2+z^2)} dx$$

with $\hat{R}'(x) = \cos\theta\,\hat{z} + \sin\theta\,\hat{x}$

O.K., E_x at point P equals 0.

Also, $\cos\theta = z / \sqrt{x^2+z^2}$

$$\Rightarrow \frac{\rho_l z}{4\pi\epsilon}\int_{-L}^{L} \frac{dx}{(x^2+z^2)^{3/2}}\,\hat{z}$$

$$= \vec{E}(0,z) = \frac{1}{4\pi\epsilon}\cdot\frac{2\rho_l L}{z\sqrt{z^2+L^2}}\,\hat{z}$$

Special Case $z \gg L$

$$\Rightarrow \vec{E}(0,z) \approx \frac{q\,\hat{z}}{4\pi\epsilon z^2}$$

$q = 2\rho_l L$

(pseudo point charge)

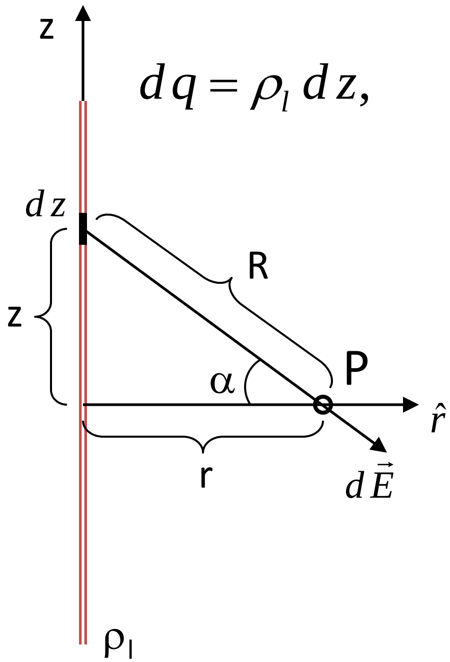
Special Case $L \to \infty$

$$\Rightarrow \vec{E}(z) = \frac{1}{4\pi\epsilon}\cdot\frac{2\rho_l}{z}$$

("infinitely" long wire)

Examples:

2. Derive an expression about the E-field due to an infinitely thin and long uniformly charged rod.



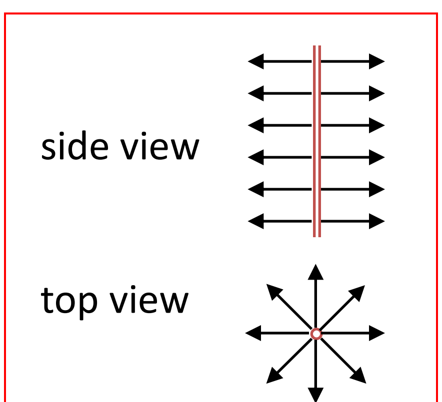
$$dq = \rho_l\, dz, \qquad \vec{E} = \int_{-\infty}^{\infty} d\vec{E}$$

$$d\vec{E} = \frac{\rho_l\, dz}{4\pi\varepsilon\left(z^2 + r^2\right)}\left(\hat{r}\cos\alpha - \hat{z}\sin\alpha\right)$$

$$\cos\alpha = \frac{r}{R} = \frac{r}{\sqrt{z^2 + r^2}}, \qquad \sin\alpha = \frac{z}{R} = \frac{z}{\sqrt{z^2 + r^2}}$$

$$d\vec{E} = \frac{\rho_l\, dz}{4\pi\varepsilon\left(z^2 + r^2\right)^{3/2}}\left(\hat{r}\, r - \hat{z}\, z\right)$$

$$\vec{E} = \int_{-\infty}^{\infty}\left(\hat{r}\, r - \hat{z}\, z\right)\frac{\rho_l}{4\pi\varepsilon\left(z^2 + r^2\right)^{3/2}}\, dz = \hat{r}\,\frac{\rho_l}{2\pi\varepsilon r}$$



E-field lines in the case of an infinitely long positive line charge

Example 4-5: Electric Field of a Circular Disk of Charge

Find the electric field at point P with Cartesian coordinates $(0, 0, h)$ due to a circular disk of radius a and uniform charge density ρ_s residing in the x–y plane (Fig. 4-7). Also, evaluate $\mathbf{E}$ due to an infinite sheet of charge density ρ_s by letting $a \to \infty$.

Solution: Building on the expression obtained in Example 4-4 for the on-axis electric field due to a circular ring of charge, we can determine the field due to the circular disk by treating the disk as a set of concentric rings. A ring of radius r and width dr has an area $ds = 2\pi r\ dr$ and contains charge $dq = \rho_s\ ds = 2\pi\rho_s r\ dr$. Upon using this expression in Eq. (4.23) and also replacing b with r, we obtain the following expression for the field due to the ring:

$$d\mathbf{E} = \hat{\mathbf{z}}\ \frac{h}{4\pi\varepsilon_0(r^2+h^2)^{3/2}}\ (2\pi\rho_s r\ dr).$$

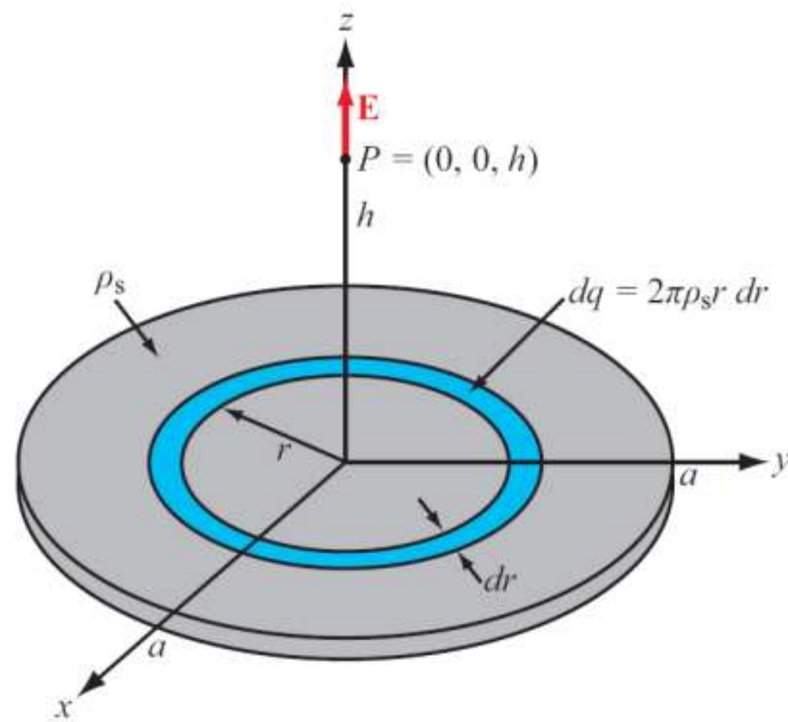


Figure 4-7: Circular disk of charge with surface charge density ρ_s. The electric field at $P = (0, 0, h)$ points along the z-direction (Example 4-5).

Cont.

Example 4-5 cont.

The total field at P is obtained by integrating the expression over the limits $r = 0$ to $r = a$:

$$\begin{aligned}\mathbf{E} &= \hat{\mathbf{z}}\,\frac{\rho_s h}{2\varepsilon_0}\int_0^a \frac{r\,dr}{(r^2+h^2)^{3/2}} \\ &= \pm\hat{\mathbf{z}}\,\frac{\rho_s}{2\varepsilon_0}\left[1-\frac{|h|}{\sqrt{a^2+h^2}}\right], \qquad (4.24)\end{aligned}$$

with the plus sign for $h > 0$ (P above the disk) and the minus sign when $h < 0$ (P below the disk).

For an infinite sheet of charge with $a = \infty$,

$$\mathbf{E} = \pm\hat{\mathbf{z}}\,\frac{\rho_s}{2\varepsilon_0} \qquad \text{(infinite sheet of charge).} \qquad (4.25)$$

We note that for an infinite sheet of charge $\mathbf{E}$ is the same at all points above the x–y plane, and a similar statement applies for points below the x-y plane.

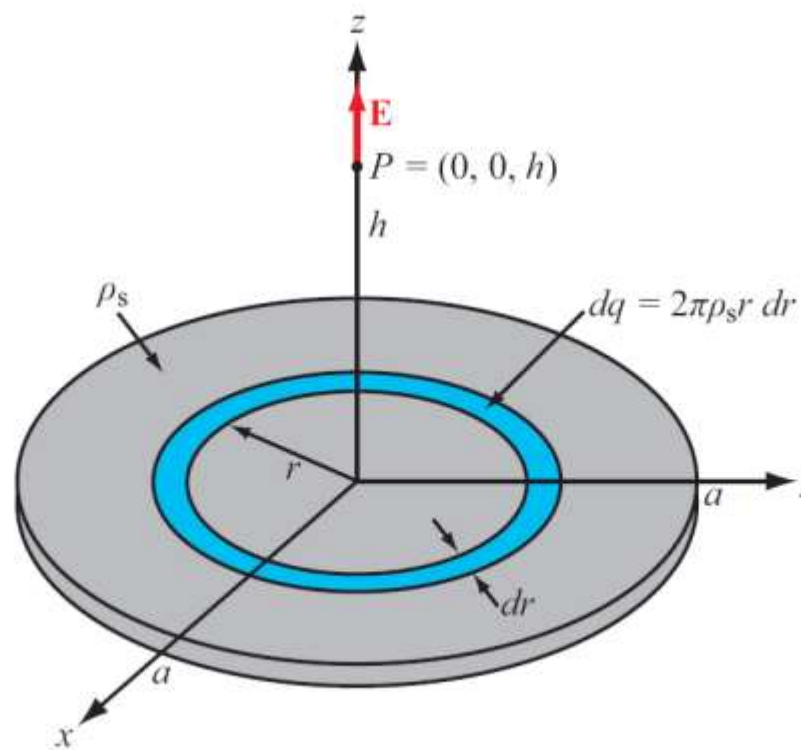
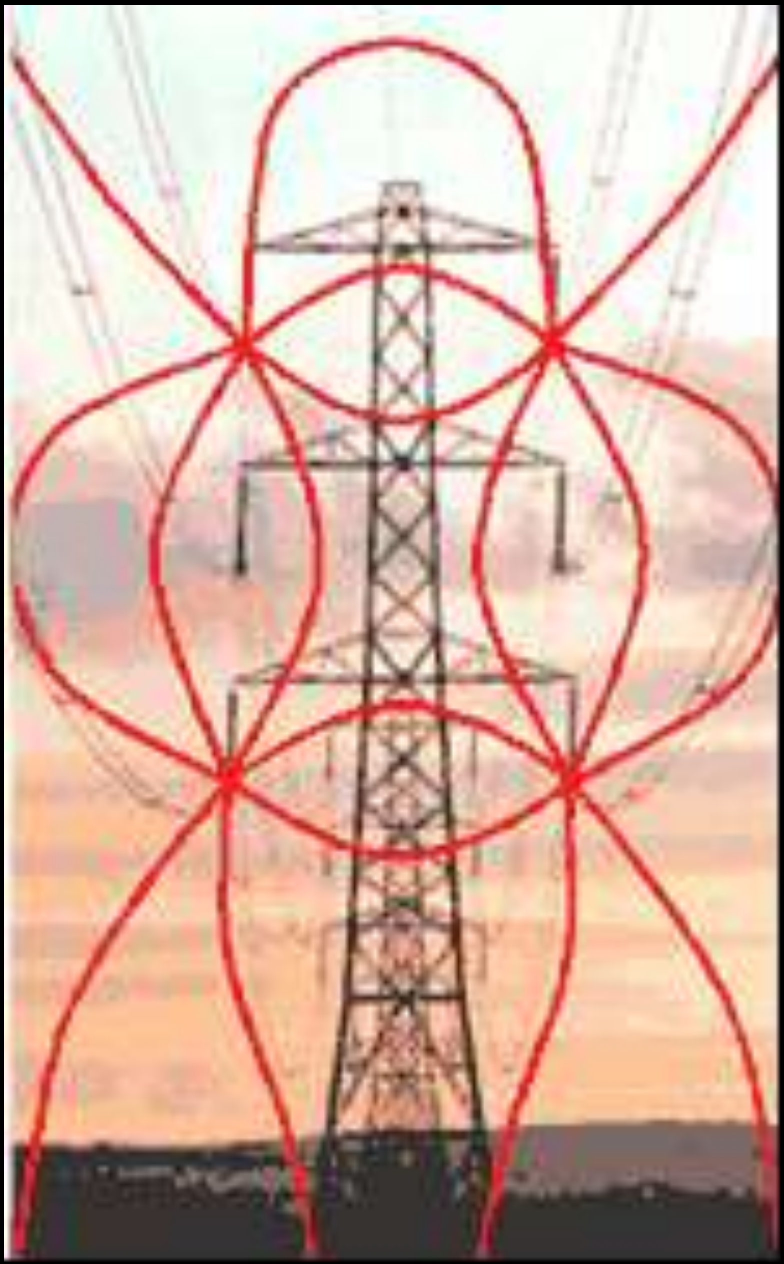


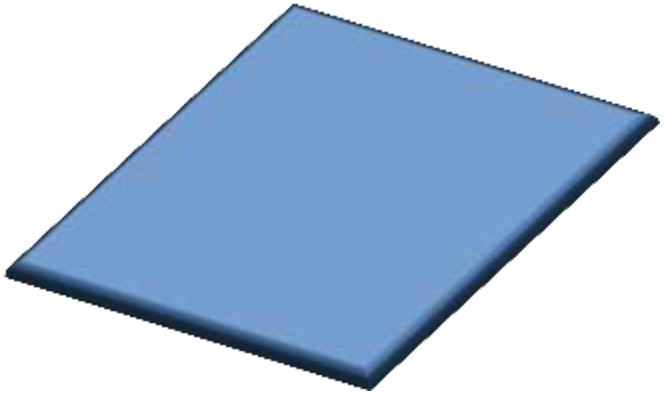
Figure 4-7: Circular disk of charge with surface charge density ρ_s. The electric field at $P = (0, 0, h)$ points along the z-direction (Example 4-5).



Note: field **lines**…

Example: Infinite sheet of charge

Find the E-field a distance z above an infinite sheet with a uniform surface charge ρ_s.



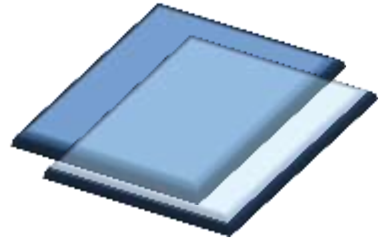
$$\vec{E} = \frac{1}{4\pi\epsilon} \int_S \frac{\hat{R}' \rho_s \, ds}{R'^2}$$

Turns out to be easier in cylindrical coords, and letting $r \to \infty$ (see text)

$$\vec{E}(z) = \frac{\rho_s}{2\epsilon} \hat{z}$$

Note: Does NOT depend on z

An infinite sheet of charge with uniform surface charge density ρ_s is located at $z = 0$ (x–y plane), and another infinite sheet with density $-\rho_s$ is located at $z = 2$ m, both in free space. Determine $\mathbf{E}$ in all regions.



for the sheet at $z = 0$,

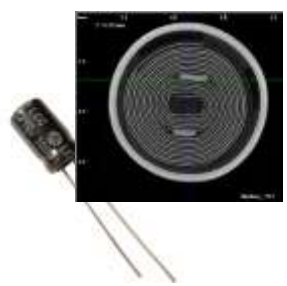
$$\mathbf{E}_1 = \begin{cases} \hat{\mathbf{z}} \dfrac{\rho_s}{2\varepsilon_0}, & \text{for } z > 0, \\ -\hat{\mathbf{z}} \dfrac{\rho_s}{2\varepsilon_0}, & \text{for } z < 0. \end{cases}$$

Similarly, for the sheet at $z = 2$ m with charge density $-\rho_s$,

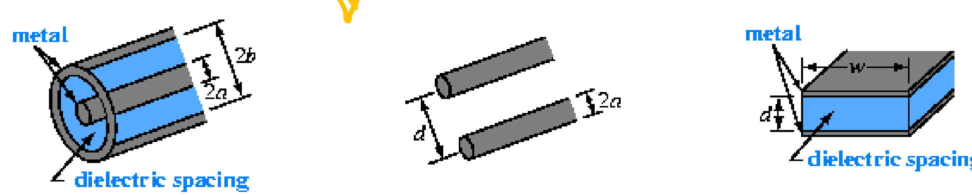
$$\mathbf{E}_2 = \begin{cases} -\hat{\mathbf{z}} \dfrac{\rho_s}{2\varepsilon_0}, & \text{for } z > 2 \text{ m}, \\ \hat{\mathbf{z}} \dfrac{\rho_s}{2\varepsilon_0}, & \text{for } z < 2 \text{ m}. \end{cases}$$

Hence,

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \begin{cases} 0, & \text{for } z < 0, \\ \hat{\mathbf{z}} \dfrac{\rho_s}{\varepsilon_0}, & \text{for } 0 < z < 2 \text{ m}, \\ 0, & \text{for } z > 2 \text{ m}. \end{cases}$$



Hmm...



(a) Coaxial line (b) Two-wire line (c) Parallel-plate line

Example 4-5: Electric Field of a Circular Disk of Charge

Find the electric field at point P with Cartesian coordinates $(0, 0, h)$ due to a circular disk of radius a and uniform charge density ρ_s residing in the x–y plane (Fig. 4-7). Also, evaluate $\mathbf{E}$ due to an infinite sheet of charge density ρ_s by letting $a \to \infty$.

Solution: Building on the expression obtained in Example 4-4 for the on-axis electric field due to a circular ring of charge, we can determine the field due to the circular disk by treating the disk as a set of concentric rings. A ring of radius r and width dr has an area $ds = 2\pi r\; dr$ and contains charge $dq = \rho_s\; ds = 2\pi\rho_s r\; dr$. Upon using this expression in Eq. (4.23) and also replacing b with r, we obtain the following expression for the field due to the ring:

$$d\mathbf{E} = \hat{\mathbf{z}}\, \frac{h}{4\pi\varepsilon_0 (r^2 + h^2)^{3/2}}\, (2\pi\rho_s r\; dr).$$

The total field at P is obtained by integrating the expression over the limits $r = 0$ to $r = a$:

$$\begin{aligned} \mathbf{E} &= \hat{\mathbf{z}}\, \frac{\rho_s h}{2\varepsilon_0} \int_0^a \frac{r\; dr}{(r^2 + h^2)^{3/2}} \\ &= \pm\hat{\mathbf{z}}\, \frac{\rho_s}{2\varepsilon_0} \left[1 - \frac{|h|}{\sqrt{a^2 + h^2}}\right], \end{aligned} \tag{4.24}$$

with the plus sign for $h > 0$ (P above the disk) and the minus sign when $h < 0$ (P below the disk).

For an infinite sheet of charge with $a = \infty$,

$$\mathbf{E} = \pm\hat{\mathbf{z}}\, \frac{\rho_s}{2\varepsilon_0} \qquad \text{(infinite sheet of charge).} \tag{4.25}$$

We note that for an infinite sheet of charge $\mathbf{E}$ is the same at all points above the x–y plane, and a similar statement applies for points below the x-y plane.

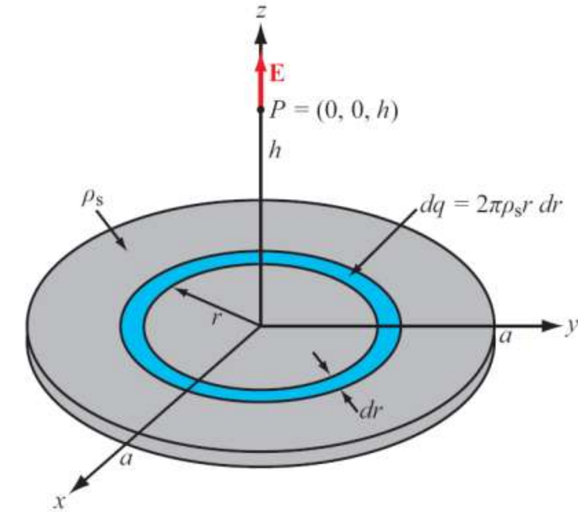


Figure 4-7: Circular disk of charge with surface charge density ρ_s. The electric field at $P = (0, 0, h)$ points along the z-direction (Example 4-5).

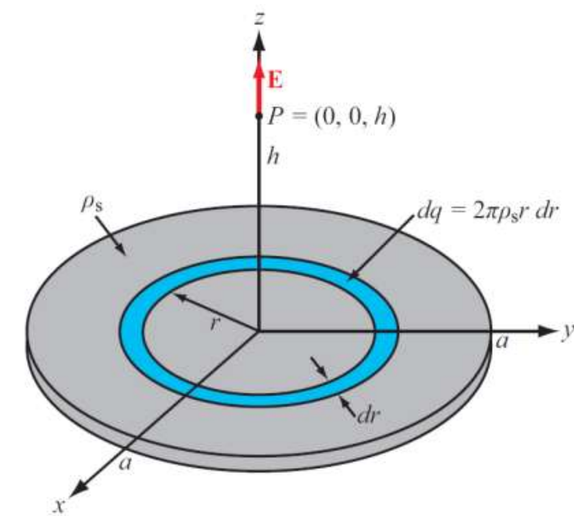


Figure 4-7: Circular disk of charge with surface charge density ρ_s. The electric field at $P = (0, 0, h)$ points along the z-direction (Example 4-5).

Example: charge on a sphere

Find the E-field a distance *z* above a sphere with a uniform surface charge ρ_s.

Special Case: E-field at a height of 1 m ?

$$\vec{E}(z) \cong \frac{\rho_s}{2\epsilon}\hat{R}$$

Looks like an infinite sheet.

Special Case: E-field as seen from Pluto ?

$$\vec{E} \cong \hat{R}\frac{Q}{4\pi\epsilon R^2}$$

Looks like a point charge

Practical Implementation

What if the charge distribution is not uniform?
What if the charged volume/surface/line is irregular ?

$$\vec{E} = \frac{1}{4\pi\epsilon}\int_V \hat{R}'\frac{\rho_v\,dv}{R'^2}$$

and similar for surface & line charges...

NUMERICAL Integration

or...

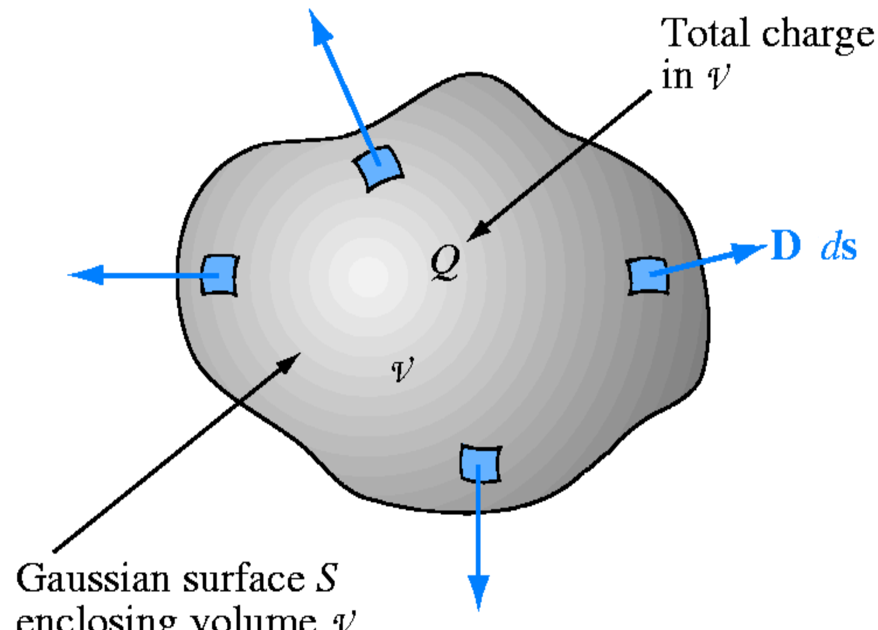
Q = total charge in volume

$$\int_{volume} \nabla\cdot\mathbf{D}\, dv = \overbrace{\int_{volume} \rho_v\, dv}^{Q}$$

$\oint_S \vec{D}\cdot d\vec{s}$ ⇐ Divergence Theorem!

Both are Gauss' Law!

$$\boxed{\oint_S \mathbf{D}\cdot d\mathbf{s} = Q}$$



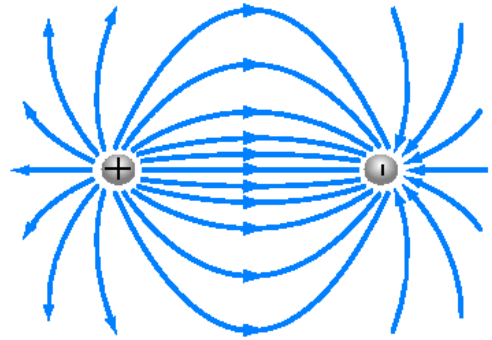
Physical meaning of Gauss' electric law:
Electric charges are sources of electric field.

http://www.youtube.com/watch?v=whv_d-fBCg0

Physical meaning of Gauss' magnetic law:
Magnetic charges do not exist.

Physical meaning of Ampere's law:
Electric currents produce magnetic fields.

Electric Fields from Distributed Charge

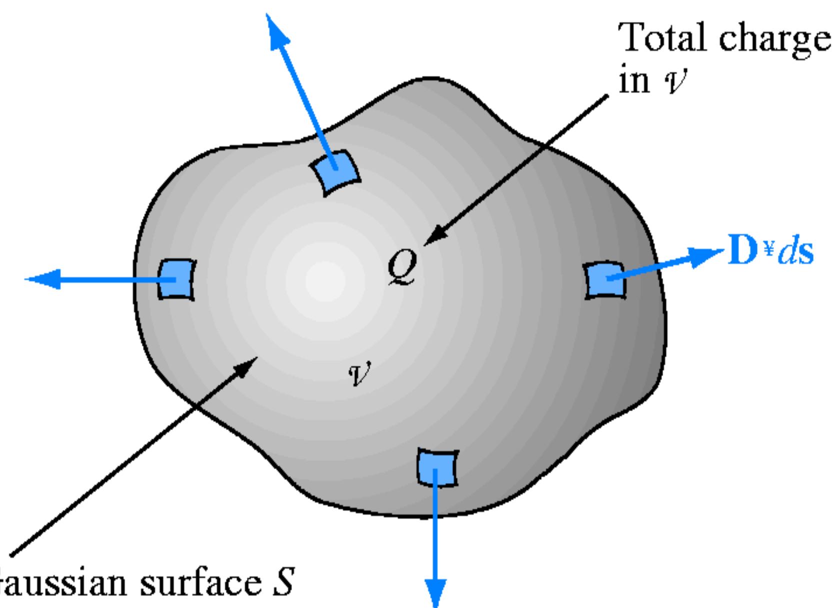


Field lines can **not** cross ☺

$$\vec{E} = \frac{1}{4\pi\epsilon}\int_V \hat{R}' \frac{\rho_v \, dv}{R'^2} \qquad \vec{E} = \frac{1}{4\pi\epsilon}\int_S \hat{R}' \frac{\rho_s \, ds}{R'^2} \qquad \vec{E} = \frac{1}{4\pi\epsilon}\int_\ell \hat{R}' \frac{\rho_\ell \, d\ell}{R'^2}$$

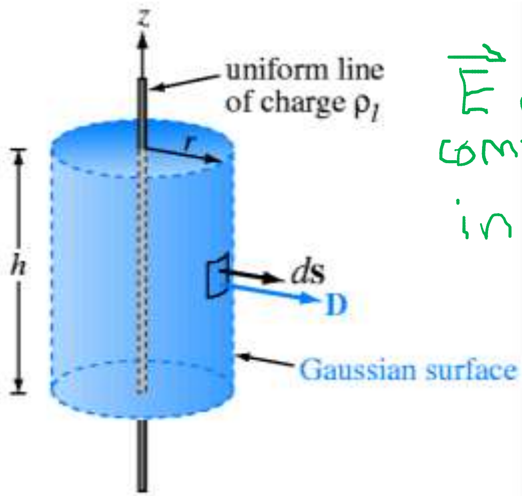
Gauss' Law

$$\boxed{\oint_S \mathbf{D}\cdot d\mathbf{s} = Q}$$



Gauss' Law is useful for *some* E-field calculations:

1. Spherical Symmetry: make Gaussian surface a concentric sphere
2. Cylindrical Symmetry: make Gaussian surface a coaxial cylinder
3. Plane symmetry: make Gaussian surface a "pillbox" which straddles the surface.



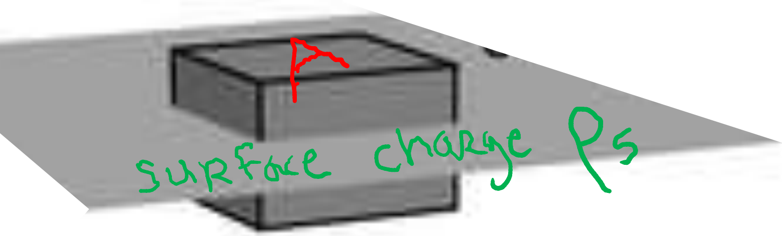
$\vec{E}$ only has components in $\hat{r}$ direction.

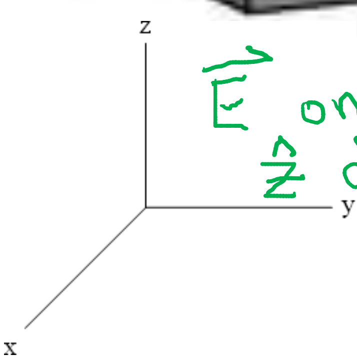
$$\oint_S \vec{E} \cdot d\vec{s} = \frac{Q}{\epsilon} \leftarrow h\rho_\ell$$

$$E_r(r) \times 2\pi r \times h$$

$$\Rightarrow \quad \vec{E}(r) = \frac{\rho_\ell}{2\pi\epsilon r}\,\hat{r} \quad \frac{V}{m}$$

as before





$\vec{E}$ only in $\hat{z}$ direction.

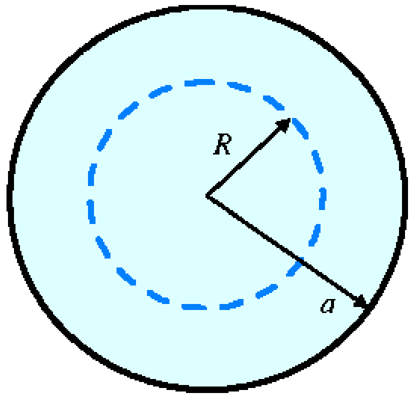
$$\oint_S \vec{E} \cdot d\vec{s} = \frac{Q}{\epsilon} \leftarrow A\rho_s$$

$$|E_z| \times A \times 2$$

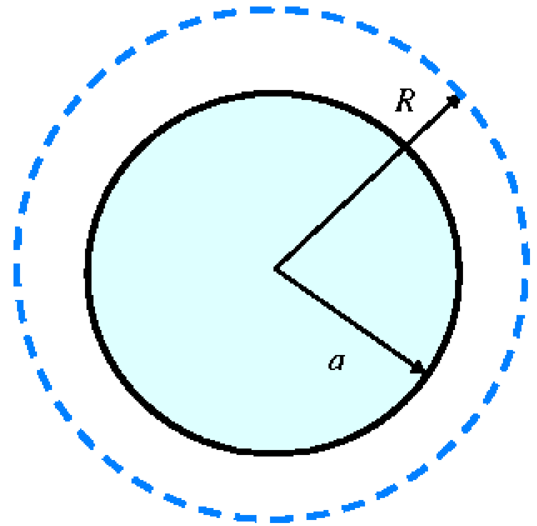
$$\Rightarrow \quad \vec{E} = \begin{cases} \rho_s/2\epsilon\ \hat{z}, & z > 0 \\ -\rho_s/2\epsilon\ \hat{z}, & z < 0 \end{cases}$$

again, as before

A spherical volume of radius a contains a uniform volume charge density ρ_v. Use Gauss's law to determine **D** for (a) $R \leq a$ and (b) $R \geq a$.



$R < a$



$R > a$

For $R \leq a$,

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \oint_S D_r \, ds = D_r(4\pi R^2)$$

Q within a sphere of radius R is

$$Q = \frac{4}{3}\pi R^3 \rho_v$$

Hence,

$$4\pi R^2 D_R = \frac{4}{3}\pi R^3 \rho_v$$

$$D_r = \frac{\rho_v R}{3}, \qquad \mathbf{D} = \hat{\mathbf{R}} D_r = \hat{\mathbf{R}} \frac{\rho_v R}{3}, \qquad R \leq a.$$

For $R \geq a$, total charge in sphere is

$$Q = \frac{4}{3}\pi a^3 \rho_v,$$

$$4\pi R^2 D_R = \frac{4}{3}\pi a^3 \rho_v,$$

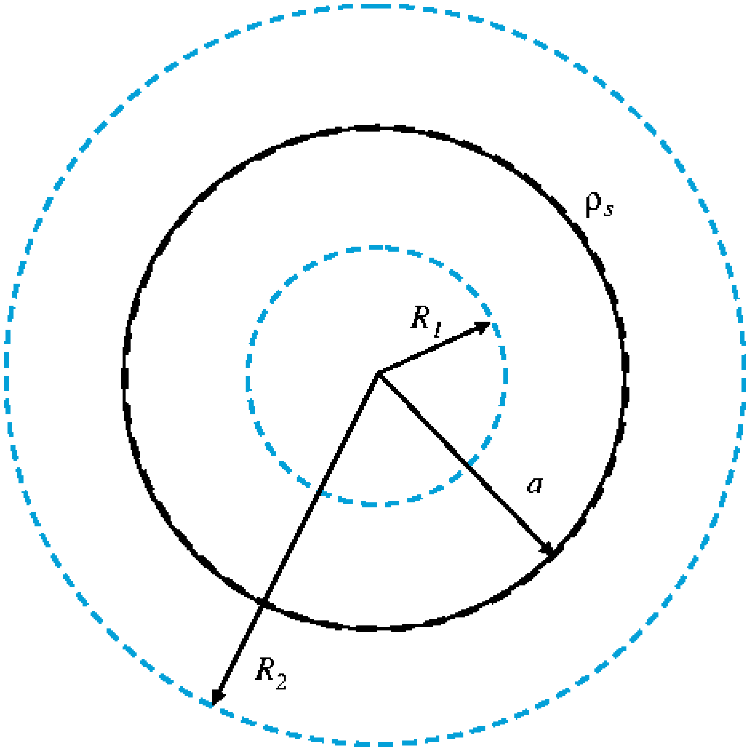
$$\mathbf{D} = \hat{\mathbf{R}} D_r = \hat{\mathbf{R}} \frac{\rho_v a^3}{3R^2}, \qquad R \geq a.$$

Q: what if the overall volume is **not** a sphere, but the inner Gaussian surface still is?

Same as if charge had all been concentrated at center !!!!

A thin spherical shell of radius a carries a uniform surface charge density ρ_s. Use Gauss's law to determine **E**.

$$\oint_S \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon}$$



$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$$

Symmetry suggests that **D** is radial in direction. Hence,

$$\mathbf{D} = \hat{\mathbf{R}} D_R$$

$$d\mathbf{s} = \hat{\mathbf{R}}\, ds$$

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \oint_S D_R\, ds = D_R(4\pi R^2) = Q$$

$$D_R = \frac{Q}{4\pi R^2}$$

- For a Gaussian surface of radius $R_1 < a$, no charge is enclosed. Hence, $Q = 0$, in which case $E = 0$.
- For a Gaussian surface of radius $R_2 > a$,

$$Q = \rho_s(4\pi a^2)$$

and

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon} = \frac{\hat{\mathbf{R}}}{\varepsilon} D_r = \frac{\hat{\mathbf{R}} Q}{4\pi\varepsilon R_2^2} = \hat{\mathbf{R}} \frac{4\pi\rho_s a^2}{4\pi\varepsilon R_2^2} = \hat{\mathbf{R}} \frac{\rho_s a^2}{\varepsilon R_2^2}.$$

Problem 4.16 Multiple charges at different locations are said to be in equilibrium if the force acting on any one of them is identical in magnitude and direction to the force acting on any of the others. Suppose we have two negative charges, one located at the origin and carrying charge $-9e$, and the other located on the positive x-axis at a distance d from the first one and carrying charge $-36e$. Determine the location, polarity and magnitude of a third charge whose placement would bring the entire system into equilibrium.

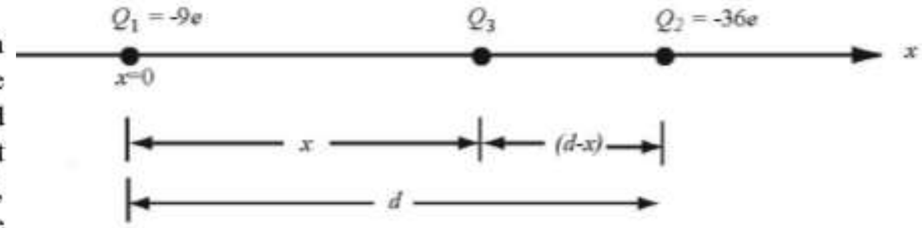


Figure P4.16: Three collinear charges.

Solution: If

$$\mathbf{F}_1 = \text{force on } Q_1,$$
$$\mathbf{F}_2 = \text{force on } Q_2,$$
$$\mathbf{F}_3 = \text{force on } Q_3,$$

then equilibrium means that

$$\mathbf{F}_1 = \mathbf{F}_2 = \mathbf{F}_3.$$

The two original charges are both negative, which mean they would repel each other. The third charge has to be positive and has to lie somewhere between them in order to counteract their repulsion force. The forces acting on charges Q_1, Q_2, and Q_3 are respectively

$$\mathbf{F}_1 = \frac{\hat{\mathbf{R}}_{21}Q_1Q_2}{4\pi\varepsilon_0 R_{21}^2} + \frac{\hat{\mathbf{R}}_{31}Q_1Q_3}{4\pi\varepsilon_0 R_{31}^2} = -\hat{\mathbf{x}}\frac{324e^2}{4\pi\varepsilon_0 d^2} + \hat{\mathbf{x}}\frac{9eQ_3}{4\pi\varepsilon_0 x^2},$$
$$\mathbf{F}_2 = \frac{\hat{\mathbf{R}}_{12}Q_1Q_2}{4\pi\varepsilon_0 R_{12}^2} + \frac{\hat{\mathbf{R}}_{32}Q_3Q_2}{4\pi\varepsilon_0 R_{32}^2} = \hat{\mathbf{x}}\frac{324e^2}{4\pi\varepsilon_0 d^2} - \hat{\mathbf{x}}\frac{36eQ_3}{4\pi\varepsilon_0 (d-x)^2},$$
$$\mathbf{F}_3 = \frac{\hat{\mathbf{R}}_{13}Q_1Q_3}{4\pi\varepsilon_0 R_{13}^2} + \frac{\hat{\mathbf{R}}_{23}Q_2Q_3}{4\pi\varepsilon_0 R_{23}^2} = -\hat{\mathbf{x}}\frac{9eQ_3}{4\pi\varepsilon_0 x^2} + \hat{\mathbf{x}}\frac{36eQ_3}{4\pi\varepsilon_0 (d-x)^2}.$$

Hence, equilibrium requires that

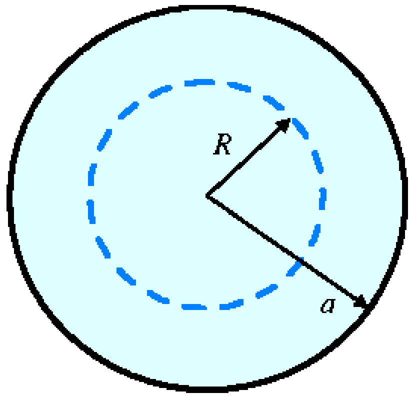
$$-\frac{324e}{d^2} + \frac{9Q_3}{x^2} = \frac{324e}{d^2} - \frac{36Q_3}{(d-x)^2} = -\frac{9Q_3}{x^2} + \frac{36Q_3}{(d-x)^2}.$$

Solution of the above equations yields

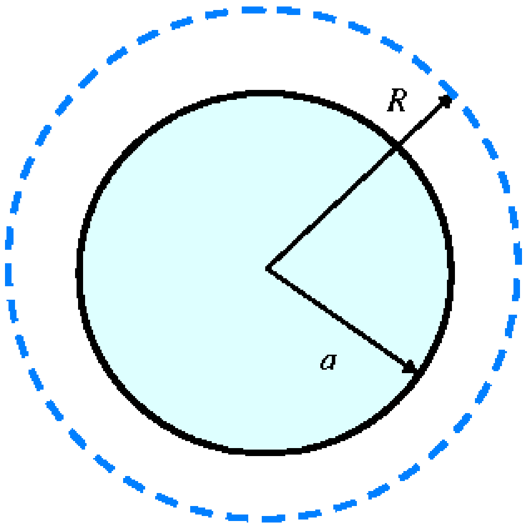
$$Q_3 = 4e, \qquad x = \frac{d}{3}.$$

"I pointed out to you the stars (the moon) and all you saw was the tip of my finger"
- Tanzania Proverb quotes

A spherical volume of radius a contains a uniform volume charge density ρ_v. Use Gauss's law to determine **D** for (a) $R \le a$ and (b) $R \ge a$.



$R < a$



$R > a$

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Hence,

$$4\pi R^2 D_R = \frac{4}{3}\pi R^3 \rho_v$$

$$D_r = \frac{\rho_v R}{3}, \qquad \mathbf{D} = \hat{\mathbf{R}} D_r = \hat{\mathbf{R}} \frac{\rho_v R}{3}, \qquad R \le a.$$

Q: what if the overall volume is **not** a sphere, but the inner Gaussian surface still is?

For $R \ge a$, total charge in sphere is

$$Q = \frac{4}{3}\pi a^3 \rho_v,$$

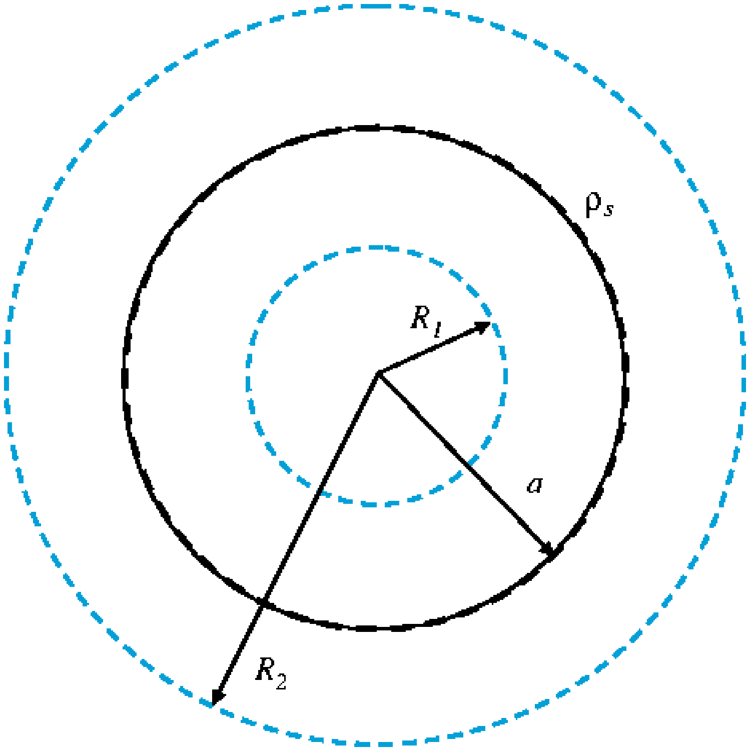
$$4\pi R^2 D_R = \frac{4}{3}\pi a^3 \rho_v,$$

$$\mathbf{D} = \hat{\mathbf{R}} D_r = \hat{\mathbf{R}} \frac{\rho_v a^3}{3R^2}, \qquad R \ge a.$$

Same as if charge had all been concentrated at center !!!!

A thin spherical shell of radius a carries a uniform surface charge density ρ_s. Use Gauss's law to determine **E**.

$$\oint_S \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon}$$



$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$$

Symmetry suggests that **D** is radial in direction. Hence,

$$\mathbf{D} = \hat{\mathbf{R}} D_R$$

$$d\mathbf{s} = \hat{\mathbf{R}}\, ds$$

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \oint_S D_R\, ds = D_R(4\pi R^2) = Q$$

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and

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon} = \frac{\hat{\mathbf{R}}}{\varepsilon} D_r = \frac{\hat{\mathbf{R}} Q}{4\pi\varepsilon R_2^2} = \hat{\mathbf{R}} \frac{4\pi\rho_s a^2}{4\pi\varepsilon R_2^2} = \hat{\mathbf{R}} \frac{\rho_s a^2}{\varepsilon R_2^2}.$$

Potential Energy

- remember *mgh* ??
- height difference, not absolute.
- path independent
- this is only an analogy ☺

Electrostatics

$$\nabla \times (\nabla f) = 0$$

$$\nabla \times \mathbf{E} = 0$$

A conservative field…

A conservative vector field can always be expressed as the **gradient** of a **scalar** field.

$$\mathbf{C}(\bar{r}) = \nabla g(\bar{r})$$

$$\vec{E} = -\nabla V$$

$$\nabla \cdot \vec{E} = \frac{\rho_v}{\epsilon} = -\nabla \cdot (\nabla V)$$

...so, what is V ??

OR

$$\nabla^2 V = \frac{-\rho_v}{\epsilon}$$

Poisson

$$\nabla^2 V = 0$$

in Regions of no free charge

Laplace

Conservative fields

$$\mathbf{C}(\bar{r}) = \nabla g(\bar{r})$$

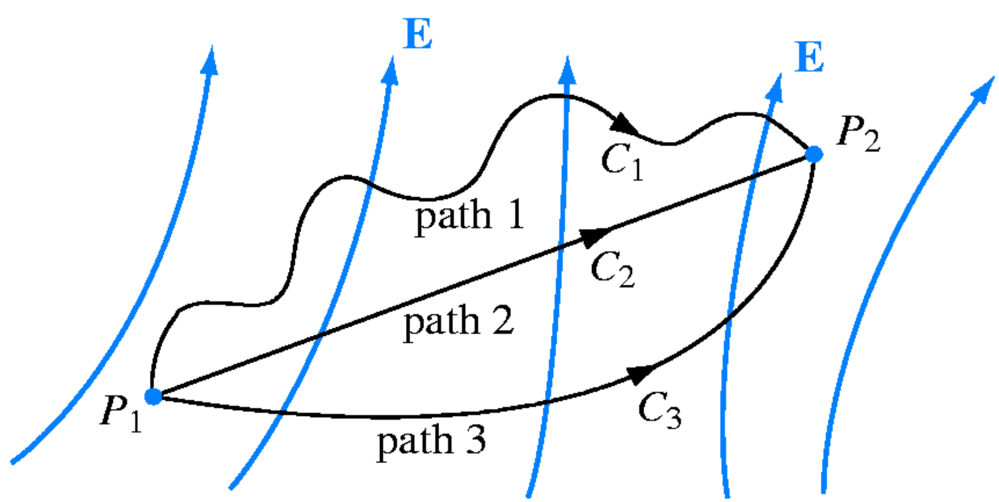
$$\vec{E} = -\nabla V$$

Integration over an **open** contour is dependent **only** on the value of scalar field $g(\bar{r})$ at the beginning and ending points of the contour (i.e., integration is **path independent**).

$$\int_C C(\bar{r})\cdot\overline{d\ell} = \int_C \nabla g(\bar{r})\cdot\overline{d\ell} = g(\bar{r}=\bar{r}_B) - g(\bar{r}=\bar{r}_A)$$

$$-\int_{P_1}^{P_2} \vec{E}\cdot d\vec{l} = V_2 - V_1$$

...remember: <u>only</u> for <u>electro-statics</u> !!!



...but, what <u>is</u> V ??

$$\vec{F} = q\vec{E}$$

$$\underset{\text{external}}{\vec{F}\cdot d\vec{l}} = \underset{\text{differential work}}{dW} = -q\vec{E}\cdot d\vec{l}$$

go against the flow

$$\frac{dW}{q} = dV = -\vec{E}\cdot d\vec{l}$$

differential electric potential

units of $\frac{J}{C}$ = VOLTS *

* and hence $\vec{E}$ has units of V/m

Problem 4.34 Given the electric field

$$\mathbf{E} = \hat{\mathbf{R}} \frac{18}{R^2} \quad \text{(V/m)},$$

find the electric potential of point A with respect to point B where A is at $+2$ m and B at -4 m, both on the z-axis.

Solution:

$$V_{AB} = V_A - V_B = -\int_B^A \mathbf{E} \cdot d\mathbf{l}.$$

Along z-direction, $\hat{\mathbf{R}} = \hat{\mathbf{z}}$ and $\mathbf{E} = \hat{\mathbf{z}} \frac{18}{z^2}$ for $z > 0$, and $\hat{\mathbf{R}} = -\hat{\mathbf{z}}$ and $\mathbf{E} = -\hat{\mathbf{z}} \frac{18}{z^2}$ for $z < 0$. Hence,

$$V_{AB} = -\int_{-4}^{2} \hat{\mathbf{R}} \frac{18}{z^2} \cdot \hat{\mathbf{z}}\, dz = -\left[\int_{-4}^{0} -\hat{\mathbf{z}} \frac{18}{z^2} \cdot \hat{\mathbf{z}}\, dz + \int_{0}^{2} \hat{\mathbf{z}} \frac{18}{z^2} \cdot \hat{\mathbf{z}}\, dz\right] = 4 \text{ V}.$$

Plus a teeny little half loop of vanishingly small length around the origin…

Electric Potential

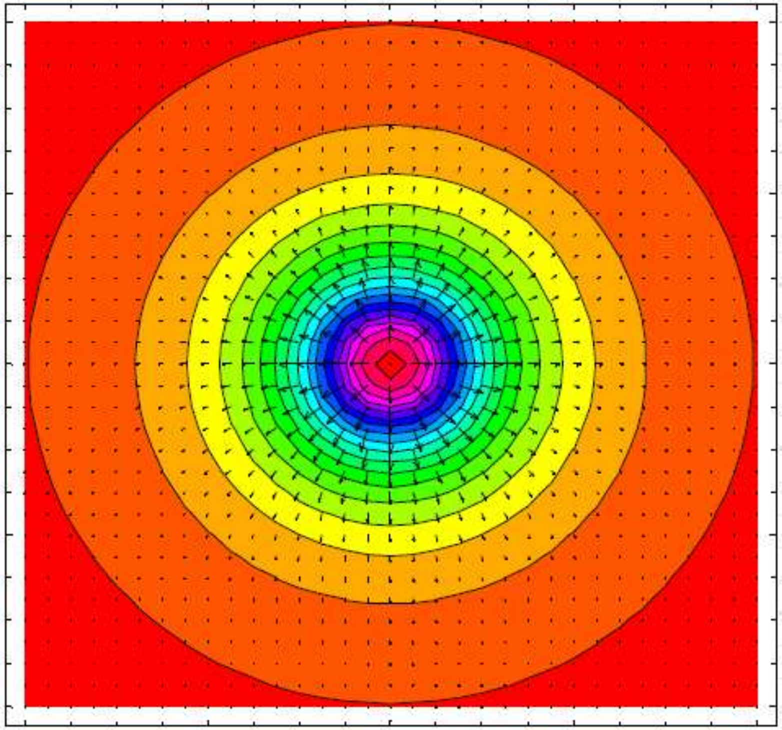
Energy per unit charge needed to bring a charged object to the point P given the background electric field

- defined between 2 points $\Rightarrow V(\vec{r}) = -\int_{\infty}^{P} \vec{E} \cdot d\vec{\ell}$
- Usually reference all points in space to a common ground... at infinity!

Point Charge

$$\boxed{V(\bar{r}) = \frac{Q}{4\pi\varepsilon_0 r}}$$

$$\boxed{V(\bar{r}) = \frac{Q}{4\pi\varepsilon_0 |\bar{r}-\bar{r}'|}}$$



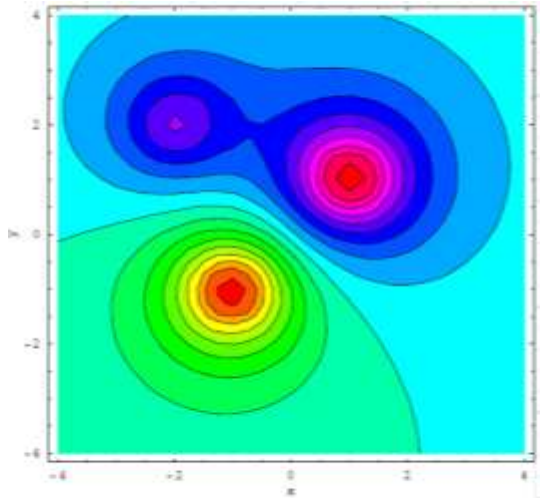
So,

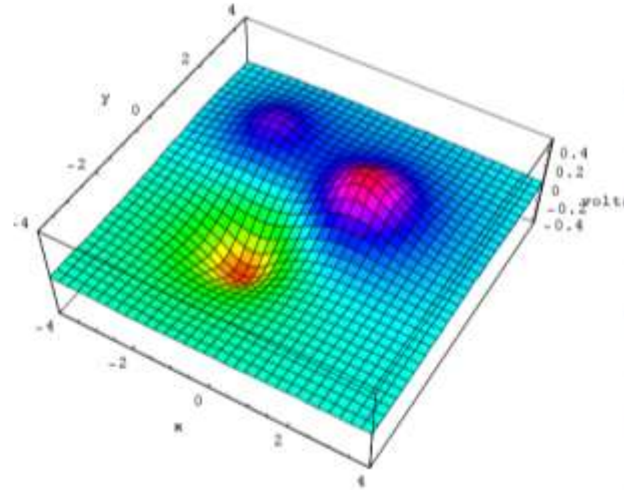
$$V(\vec{R}) = \frac{1}{4\pi\epsilon}\int_V \frac{\rho_r}{R'}\,dv$$

w/ R' = distance

and

$$\vec{E} = -\nabla V$$

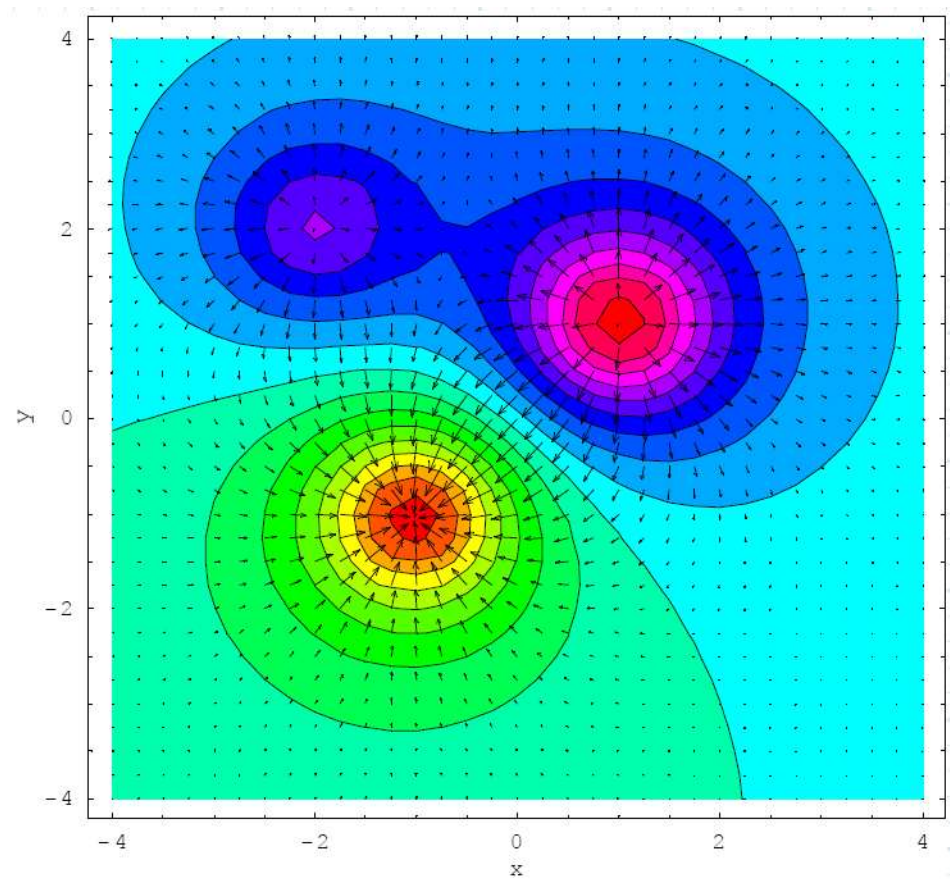




$$V = \frac{1}{4\pi\varepsilon}\int_{V'} \frac{\rho_v}{R'}\,dV' \quad \text{(volume distribution)},$$

$$V = \frac{1}{4\pi\varepsilon}\int_{S'} \frac{\rho_s}{R'}\,ds' \quad \text{(surface distribution)},$$

$$V = \frac{1}{4\pi\varepsilon}\int_{l'} \frac{\rho_\ell}{R'}\,dl' \quad \text{(line distribution)}.$$



Example 4-7: Electric Field of an Electric Dipole

Solution: To simplify the derivation, we align the dipole along the z-axis and center it at the origin [Fig. 4-13(a)]. For the two charges shown in Fig. 4-13(a), application of Eq. (4.47) gives

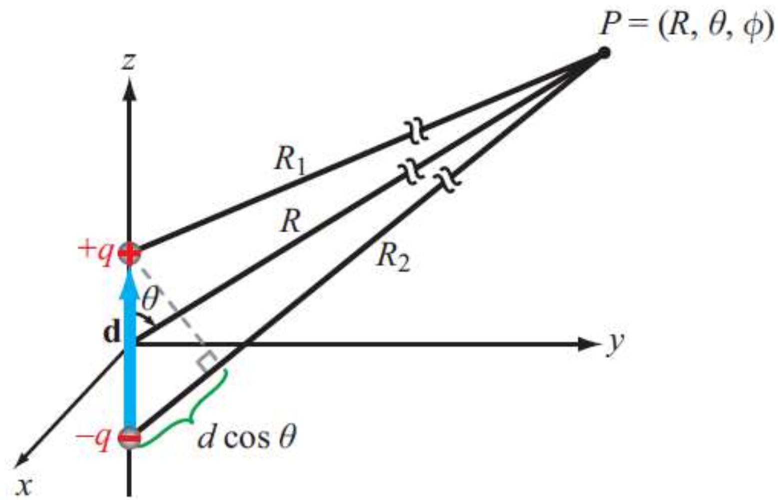
$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{R_1} + \frac{-q}{R_2}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{R_2 - R_1}{R_1 R_2}\right).$$

Since $d \ll R$, the lines labeled R_1 and R_2 in Fig. 4-13(a) are approximately parallel to each other, in which case the following approximations apply:

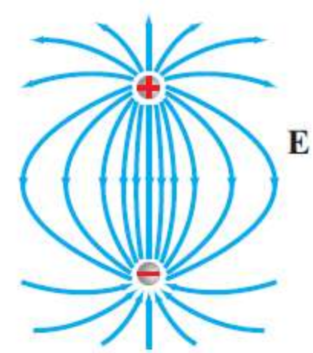
$$R_2 - R_1 \simeq d\cos\theta, \qquad R_1 R_2 \simeq R^2.$$

Hence,

$$V = \frac{qd\cos\theta}{4\pi\varepsilon_0 R^2}. \tag{4.52}$$



(a) Electric dipole



(b) Electric-field pattern

Cont.

Example 4-7: Electric Field of an Electric Dipole (cont.)

$$qd\cos\theta = q\mathbf{d}\cdot\hat{\mathbf{R}} = \mathbf{p}\cdot\hat{\mathbf{R}},$$

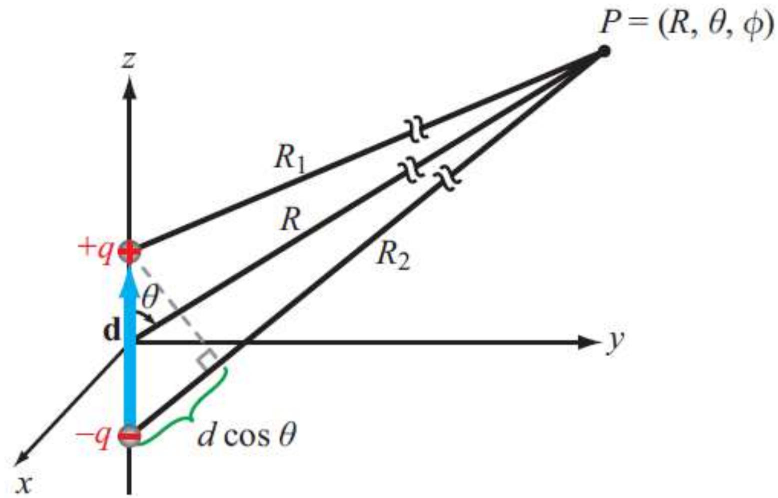
where $\mathbf{p} = q\mathbf{d}$ is called the ***dipole moment***. Using Eq. (4.53) in Eq. (4.52) then gives

$$\boxed{V = \frac{\mathbf{p}\cdot\hat{\mathbf{R}}}{4\pi\varepsilon_0 R^2} \qquad \text{(electric dipole).}} \qquad (4.54)$$

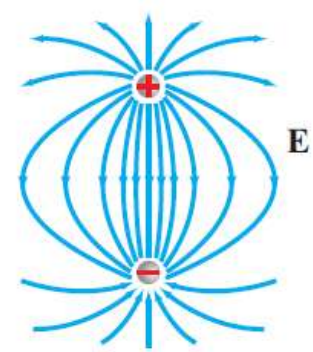
In spherical coordinates, Eq. (4.51) is given by

$$\mathbf{E} = -\nabla V = -\left(\hat{\mathbf{R}}\frac{\partial V}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\right), \qquad (4.55)$$

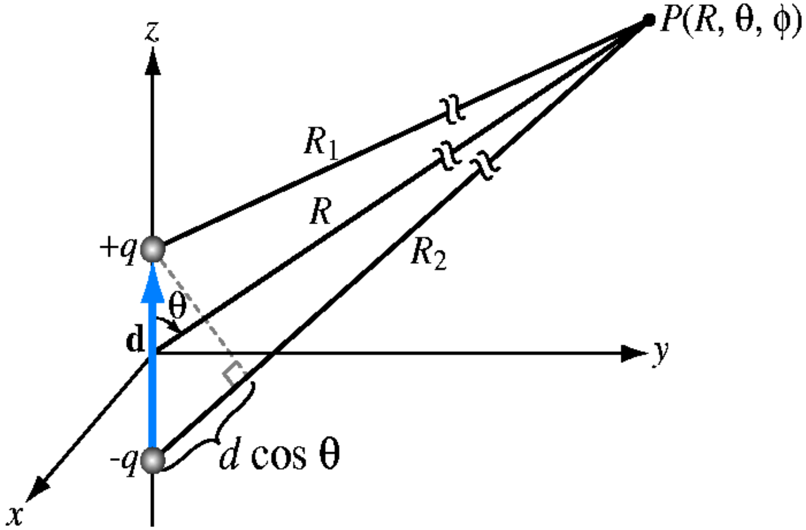
$$\boxed{\mathbf{E} = \frac{qd}{4\pi\varepsilon_0 R^3}(\hat{\mathbf{R}}\,2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta) \quad \text{(V/m).}}$$



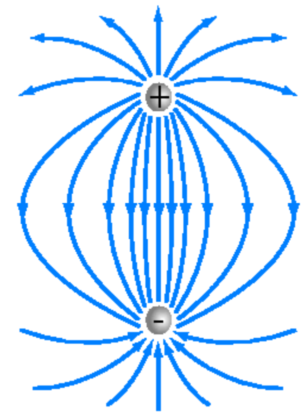
(a) Electric dipole



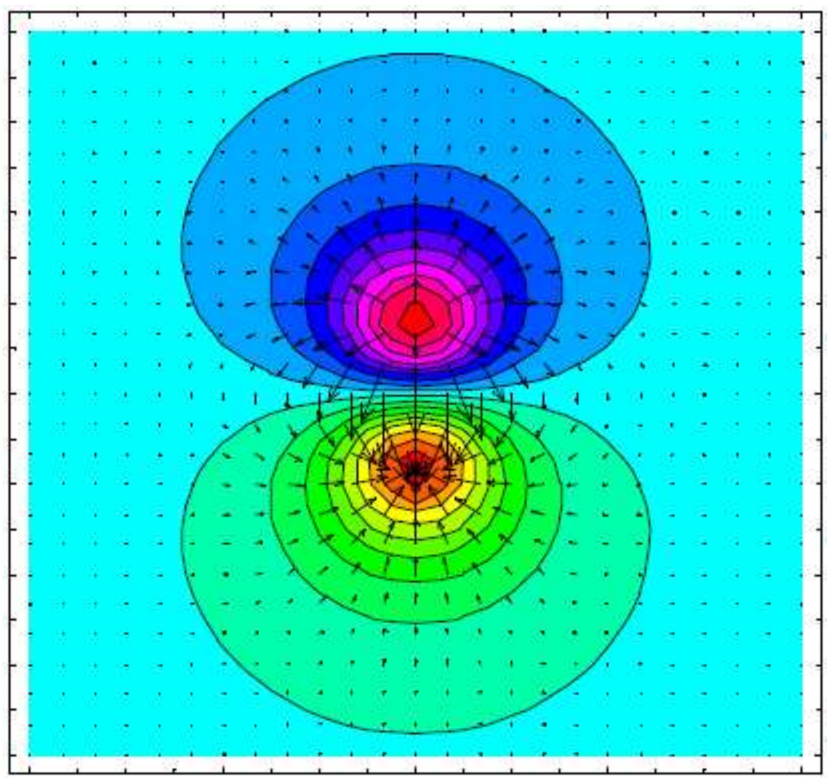
(b) Electric-field pattern



(a) Electric dipole



(b) Electric-field pattern



Poisson's & Laplace's Equations

With $\mathbf{D} = \varepsilon\mathbf{E}$, the differential form of Gauss's law given by Eq. (4.26) may be cast as

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon} . \qquad (4.57)$$

Inserting Eq. (4.51) in Eq. (4.57) gives

$$\nabla \cdot (\nabla V) = -\frac{\rho_v}{\varepsilon} . \qquad (4.58)$$

Given Eq. (3.110) for the Laplacian of a scalar function V,

$$\nabla^2 V = \nabla \cdot (\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} , \qquad (4.59)$$

Eq. (4.58) can be cast in the abbreviated form

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \qquad \text{(Poisson's equation).} \qquad (4.60)$$

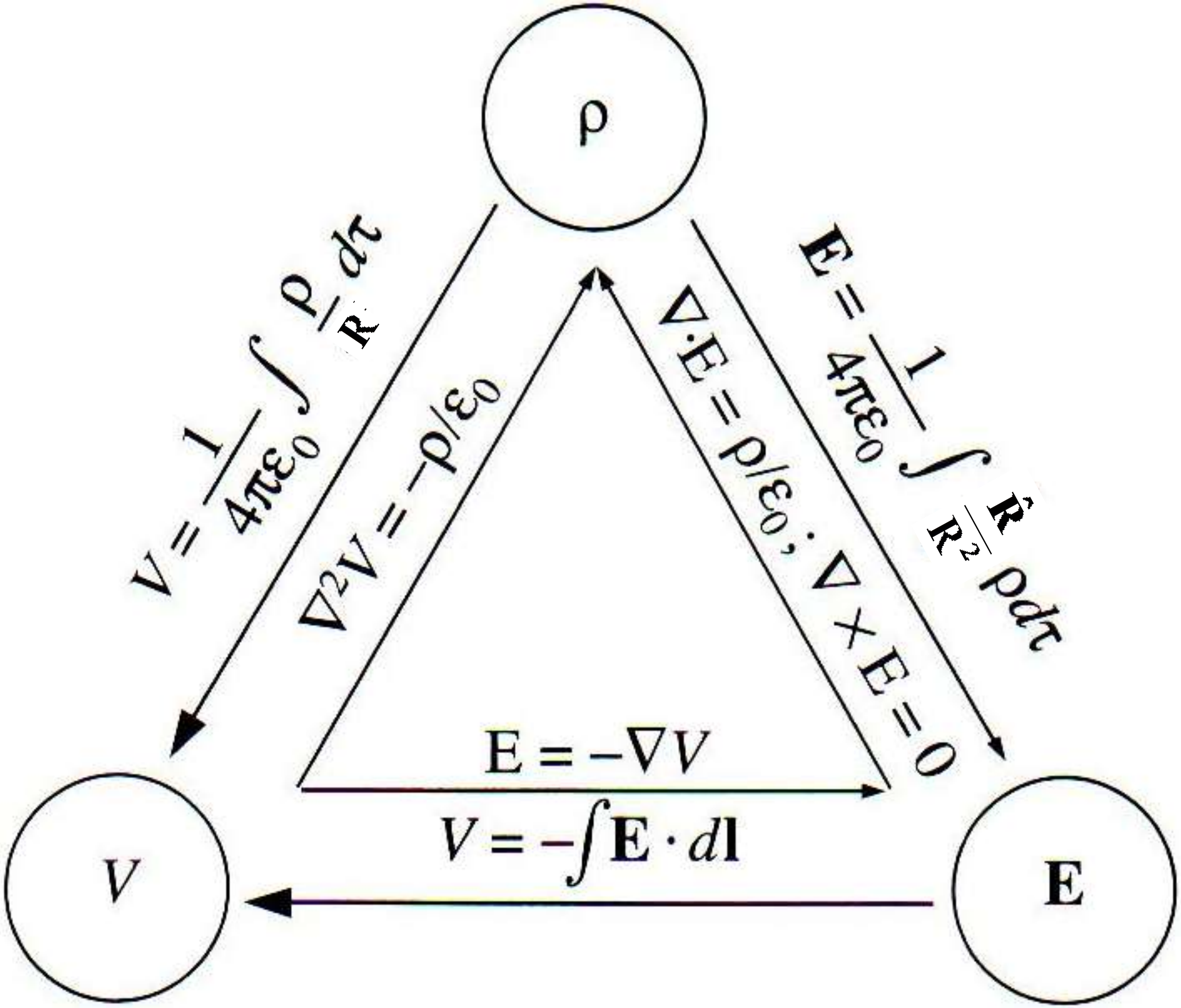
This is known as ***Poisson's equation***. For a volume $\mathcal{V}'$ containing a volume charge density distribution ρ_v, the solution for V derived previously and expressed by Eq. (4.48a) as

$$V = \frac{1}{4\pi\varepsilon} \int_{\mathcal{V}'} \frac{\rho_v}{R'} \, d\mathcal{V}' \qquad (4.61)$$

In the absence of charges:

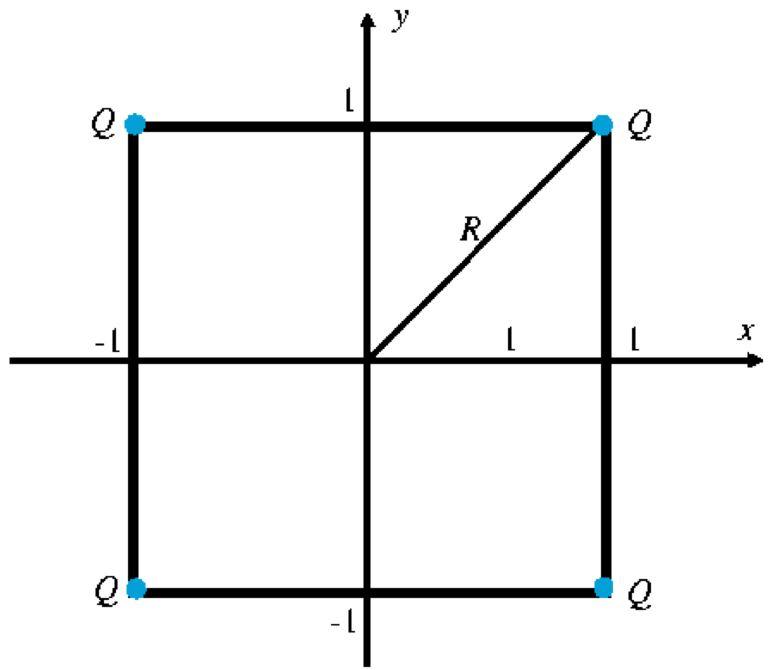
$$\nabla^2 V = 0 \qquad \text{(Laplace's equation),}$$

ElectroStatics: a Summary



http://www.falstad.com/emstatic/

Exercise 4.10 Determine the electric potential at the origin in free space due to four charges of 20 μC each located at the corners of a square in the x–y plane and whose center is at the origin. The square has sides of 2 m each.



For four identical charges all equidistant from the origin:

$$V = \frac{4Q}{4\pi\varepsilon_0 R}, \qquad R = \sqrt{2} \quad \text{(m)}$$

$$= \frac{4 \times 20 \times 10^{-6}}{4\pi\varepsilon_0 \sqrt{2}} = \frac{\sqrt{2} \times 10^{-5}}{\pi\varepsilon_0} \quad \text{(V)}.$$

What if all 4 charges were at (1,1) ???

$J = \sigma E$

Ohm's Law

$\sigma \to \infty$ perfect conductor

$\sigma \to 0$ perfect dielectric (insulator)

Material	Conductivity, σ(S/m)	Material	Conductivity, σ(S/m)
Silver	6.17×10^{7}	Fresh water	10^{-3}
Copper	5.80×10^{7}	Distilled water	2×10^{-4}
Gold	4.10×10^{7}	Dry soil	10^{-5}
Aluminum	3.54×10^{7}	Transformer oil	10^{-11}
Brass	1.57×10^{7}	Glass	10^{-12}
Bronze	10^{7}	Porcelain	2×10^{-13}
Iron	10^{7}	Rubber	10^{-15}
Seawater	4	Fused quartz	10^{-17}

σ depends on free electrons and holes!

$\vec{J} = 0$ for $\sigma = 0$

$\vec{E} = 0$ for $\sigma \to \infty$

Cool!

Exercise 4.13 The current flowing through a 100-m-long conducting wire of uniform cross section has a density of 3×10^5 (A/m^2). Find the voltage drop across the length of the wire if the wire material has a conductivity of 2×10^7 (S/m).

$$J = \sigma E$$

$$E = \frac{J}{\sigma}$$

$$V = El$$

$$= \frac{Jl}{\sigma} \quad \text{(where } l = \text{length of wire)}$$

$$= \frac{3 \times 10^5 \times 100}{2 \times 10^7} = 1.5 \quad \text{(V)}.$$

Exercise 4.14 A 50-m-long copper wire has a circular cross section with radius $r = 2$ cm. Given that the conductivity of copper is 5.8×10^7 (S/m), determine (a) the resistance R of the wire and (b) the power dissipated in the wire if the voltage across its length is 1.5 (mV).

Joule's Law

$$P = \int_V \vec{E} \cdot \vec{J} \, dv \quad \text{W}$$

(a)

$$R = \frac{l}{\sigma A} = \frac{50}{5.8 \times 10^7 \times \pi (0.02)^2} = 6.9 \times 10^{-4} \ \Omega.$$

(b)

$$P = \frac{V^2}{R} = \frac{(1.5 \times 10^{-3})^2}{6.9 \times 10^{-4}} = 3.3 \quad \text{(mW)}.$$

Example 4-9: Conductance of Coaxial Cable

The radii of the inner and outer conductors of a coaxial cable of length l are a and b, respectively (Fig. 4-15). The insulation material has conductivity σ. Obtain an expression for G', the conductance per unit length of the insulation layer.

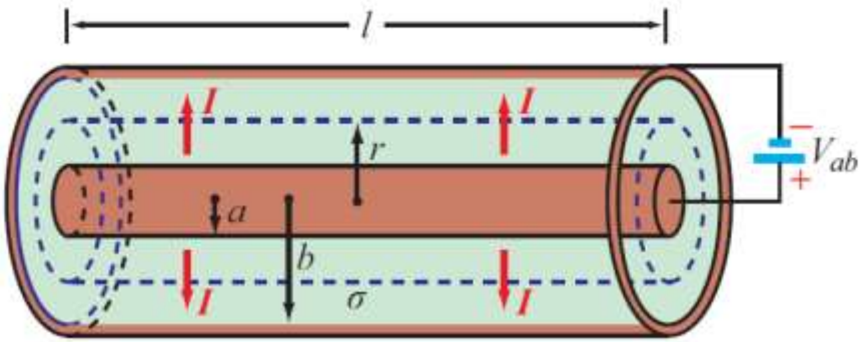
Solution: Let I be the total current flowing radially (along $\hat{\mathbf{r}}$) from the inner conductor to the outer conductor through the insulation material. At any radial distance r from the axis of the center conductor, the area through which the current flows is $A = 2\pi r l$. Hence,

$$\mathbf{J} = \hat{\mathbf{r}}\,\frac{I}{A} = \hat{\mathbf{r}}\,\frac{I}{2\pi r l}, \tag{4.73}$$

and from $\mathbf{J} = \sigma\mathbf{E}$,

$$\mathbf{E} = \hat{\mathbf{r}}\,\frac{I}{2\pi\sigma r l}. \tag{4.74}$$

In a resistor, the current flows from higher electric potential to lower potential. Hence, if $\mathbf{J}$ is in the $\hat{\mathbf{r}}$-direction, the inner



conductor must be at a higher potential than the outer conductor. Accordingly, the voltage difference between the conductors is

$$V_{ab} = -\int_b^a \mathbf{E}\cdot d\mathbf{l} = -\int_b^a \frac{I}{2\pi\sigma l}\,\frac{\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\,dr}{r} = \frac{I}{2\pi\sigma l}\ln\left(\frac{b}{a}\right). \tag{4.75}$$

The conductance per unit length is then

$$\boxed{G' = \frac{G}{l} = \frac{1}{Rl} = \frac{I}{V_{ab}l} = \frac{2\pi\sigma}{\ln(b/a)}} \quad \text{(S/m)}. \tag{4.76}$$

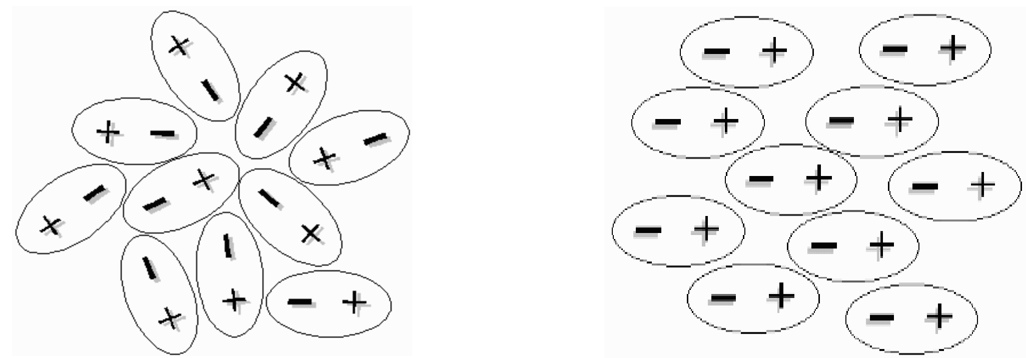
G'=0 if the insulating material is air or a perfect dielectric with zero conductivity.

Dielectric

The term was coined by Whewell in response to a request from Faraday. Whewell considered **dia-electric**, since an electric field passes through the material (Greek: dia meaning "through"), but felt that **dielectric** was easier to pronounce. The nouns **dielectric** and **insulator** are generally considered synonymous

So, charge may not be free to flow (if conductivity is zero) but it can align itself a bit with an external electric field:

$\vec{E}$



Before
(polar example)

After

$\vec{P}$

The aligned material generates its own electric polarization field:

$$\vec{P} = \chi_e \varepsilon_0 \vec{E} + \text{higher order terms}$$

??

in general

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \chi_e \varepsilon_0 \vec{E}$$

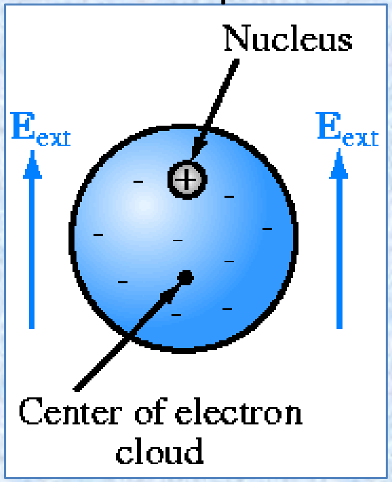
For homogeneous, isotropic & Linear media

$$\vec{D} = \varepsilon \vec{E} \quad \text{with} \quad \varepsilon = (1+\chi_e)\varepsilon_0 = \varepsilon_R \varepsilon_0$$

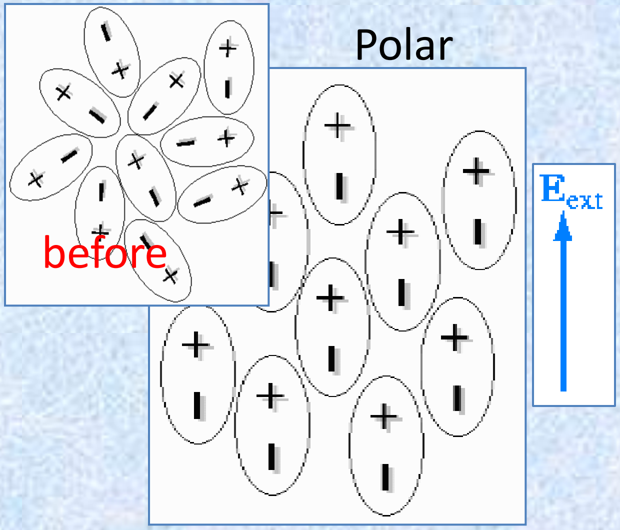
Material	Relative Permittivity,
Air	1.0
Bakelite	5
Glass	4.0-10
Mica	6
Oil	2.3
Paper	2.0-4.0
Paraffin wax	2.2
Plexiglass	3.4
Polyethylene	2.3
Polystyrene	2.55
Porcelain	5.7
Rubber	2.3-4.0
Soil (dry)	3.0-4.0
Teflon	2.1
Water (distilled)	80
Seawater	72

ϵ_R

Non-polar



Polar

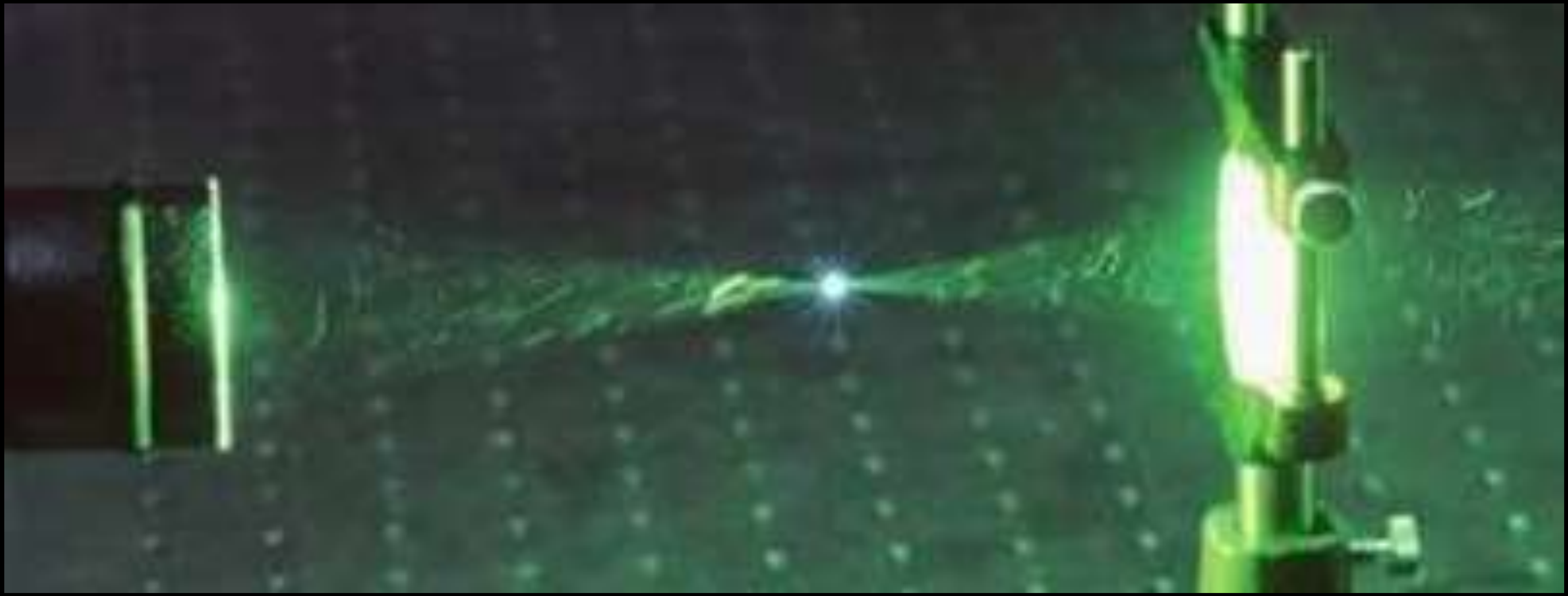


$\vec{D} = \epsilon \vec{E}$ with $\epsilon = \epsilon_R \epsilon_0$ where $\epsilon_0 \approx 8.8541818 \times 10^{-12} \frac{F}{m}$

for homogeneous, isotropic & linear media

(actually defined exactly ... we'll get to that)

Dielectric Breakdown



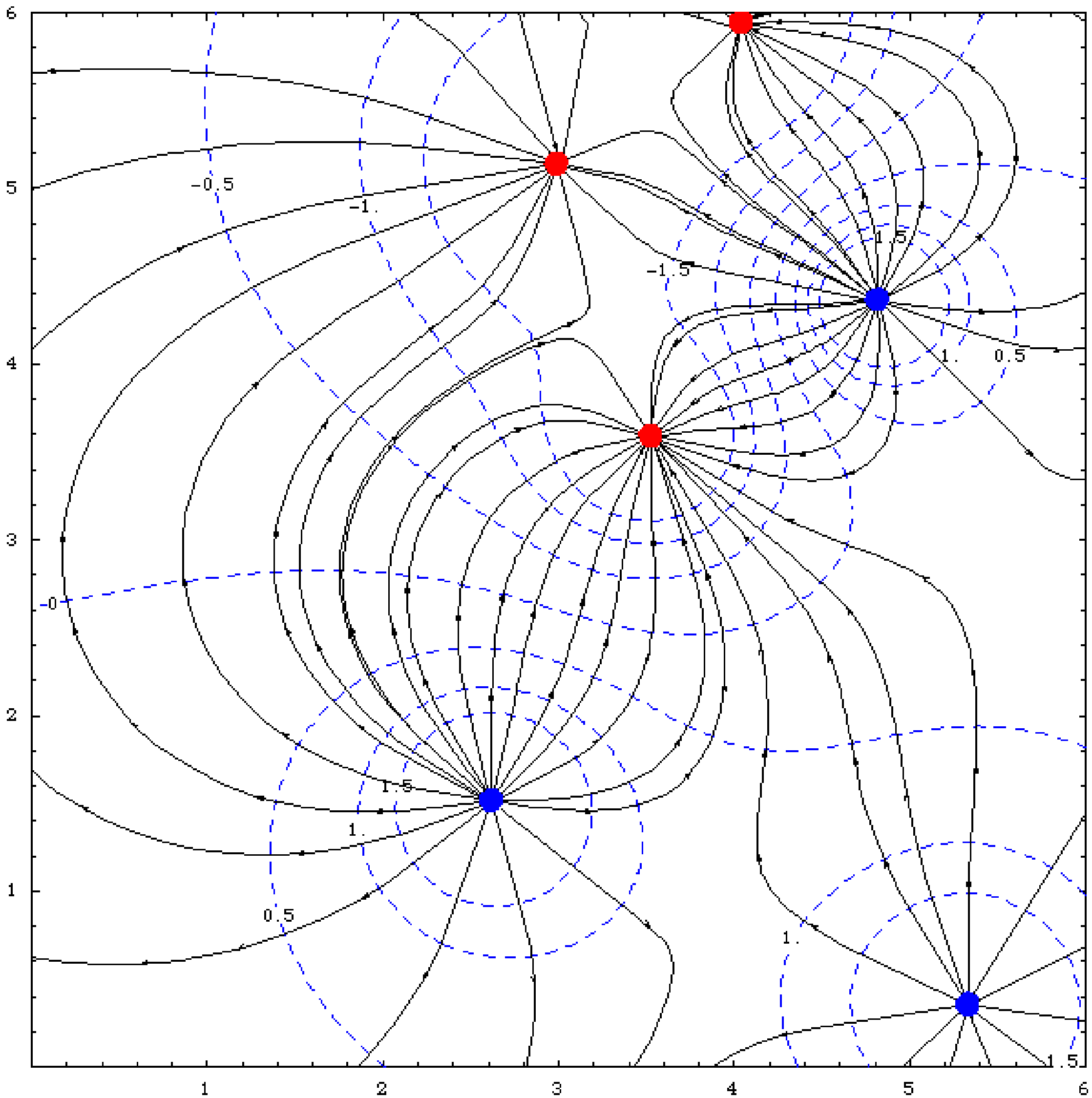
Part Dielectric, Part Conductor…

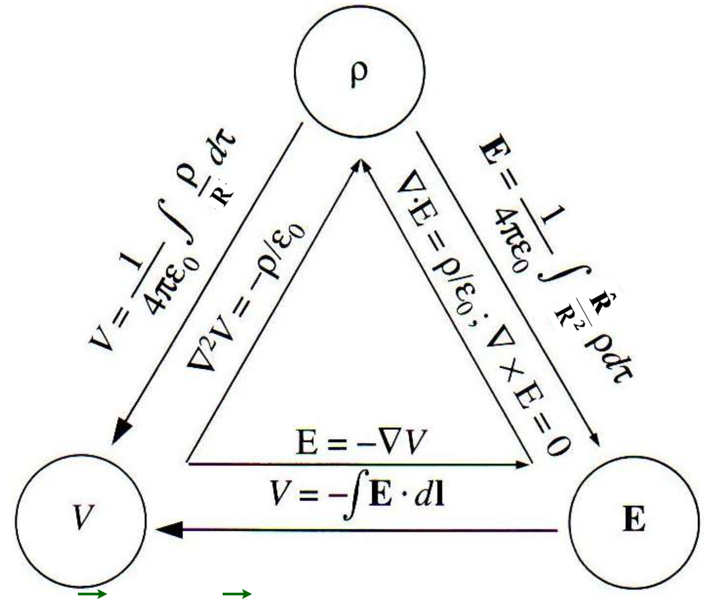
$$\vec{J} = \sigma \vec{E}$$

$$\varepsilon_C = \varepsilon_R \varepsilon_0 - j\frac{\sigma}{\omega}$$

Note: this will be for when we talk about **non**-static ☺
(but still isotropic and linear media)

$$\vec{D} = \varepsilon_c \vec{E}$$

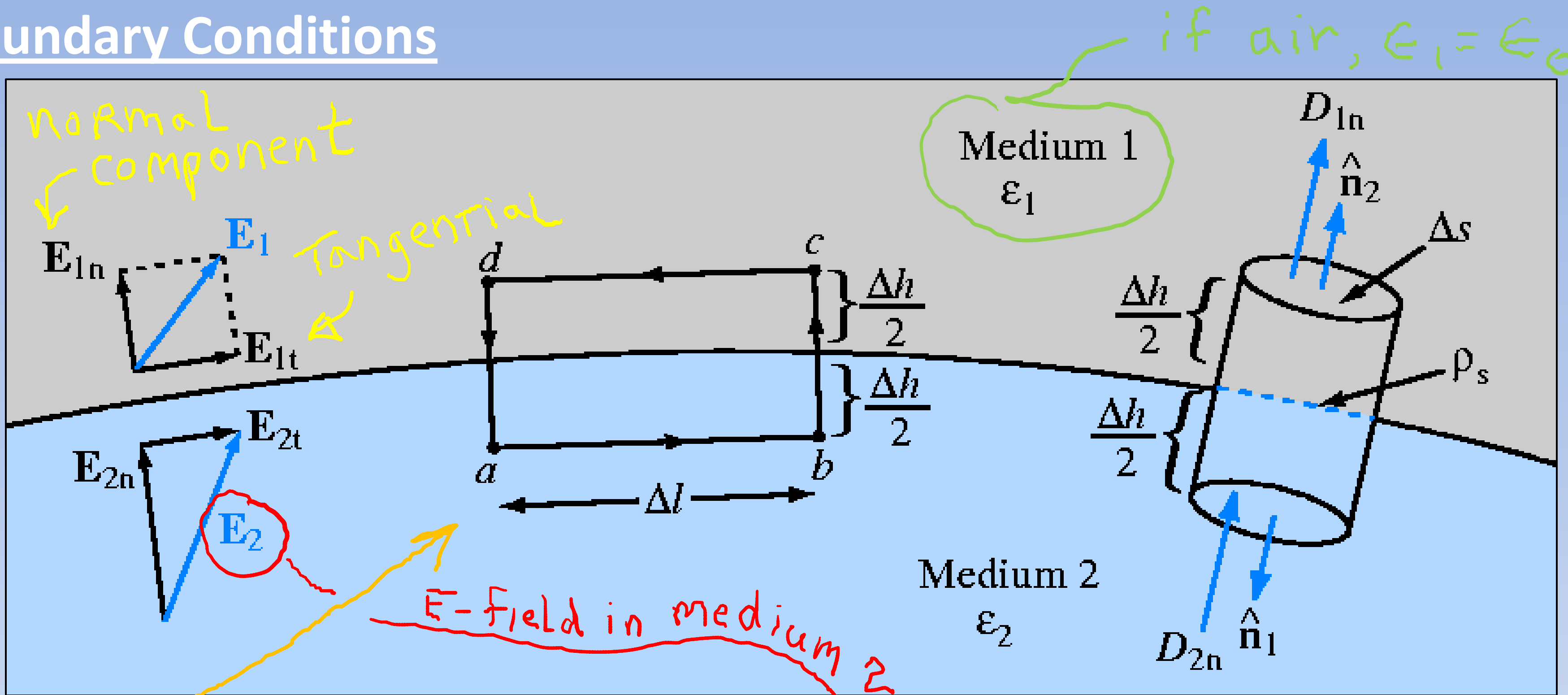




$\vec{D} = \varepsilon\vec{E}$ with $\varepsilon = \varepsilon_R\varepsilon_0$

O.K., this works fine for homogeneous material (i.e., ε and σ constant), but what if they *vary* spatially??

Boundary Conditions

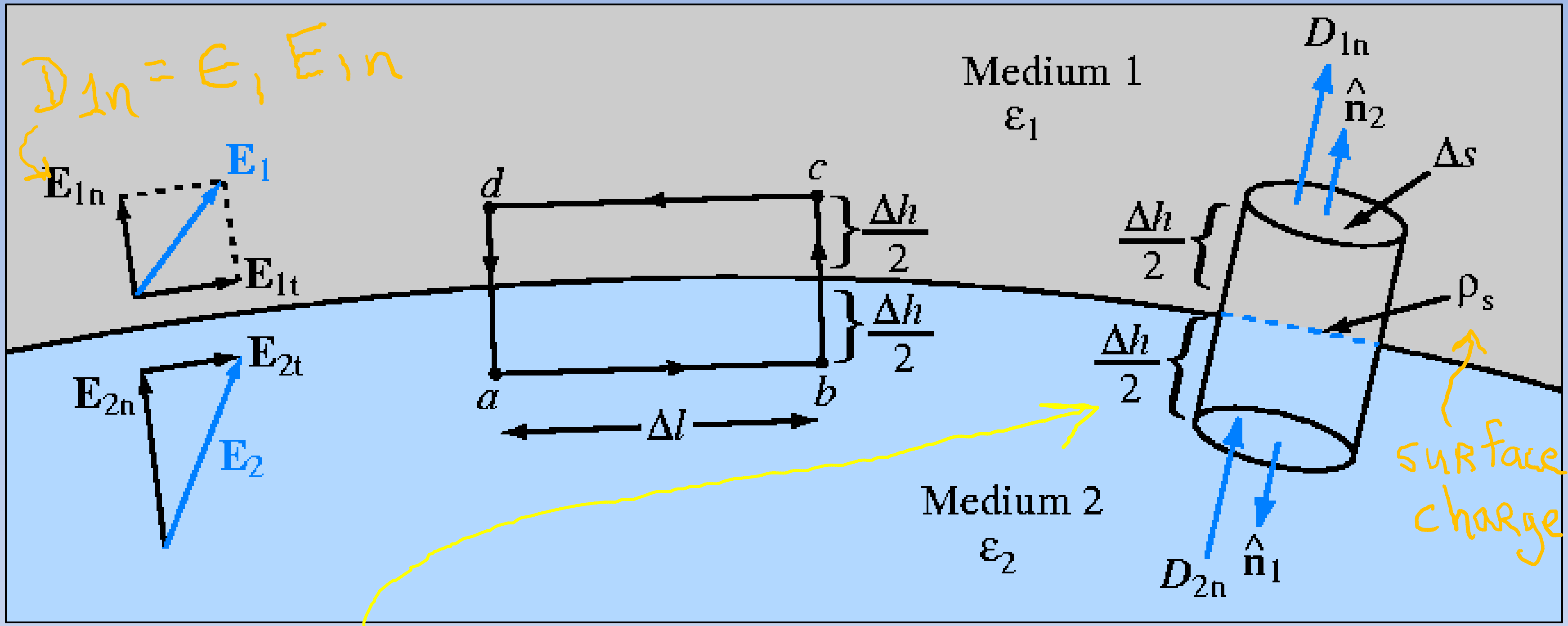


$$\underset{\text{Rectangle}}{\oint} \vec{E}\cdot d\vec{l} = \underset{\text{conserv. field}}{0} = \underbrace{\int_a^b \vec{E}_2\cdot d\vec{l}}_{|\vec{E}_{2t}|\Delta l} + \underbrace{\int_c^d \vec{E}_1\cdot d\vec{l}}_{-|\vec{E}_{1t}|\Delta l} \quad \text{if } \Delta h \to 0$$

$$\boxed{\boxed{E_{1t} = E_{2t}}}$$

Tangential component of E-field is continuous across boundary!

Boundary Conditions (more)



$$\rho_s \Delta S = \oint \vec{D} \cdot d\vec{s} = \underbrace{\int \vec{D}_1 \cdot \hat{n}_2 \, ds}_{\text{top}} + \underbrace{\int \vec{D}_2 \cdot \hat{n}_1 \, ds}_{\text{bottom}}$$

Gauss it (use $\vec{D}$ since ϵ changes)

cylinder surface

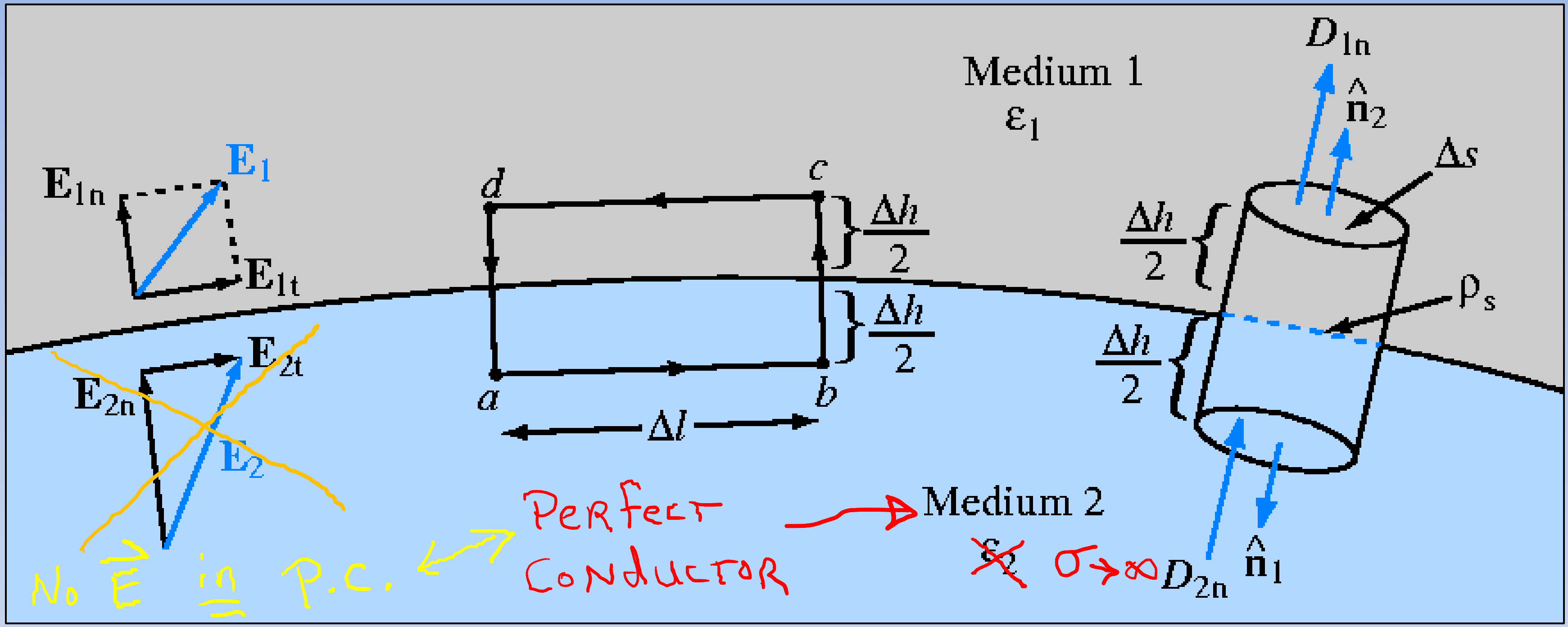
$D_{1n} \Delta S$ $\quad -D_{2n} \Delta S$

if $\Delta h \to 0$

$$\boxed{D_{1n} - D_{2n} = \rho_s}$$

Normal component of $\vec{D}$ changes abruptly at surface due to ρ_s

Boundary Conditions (*yet* more!!!)

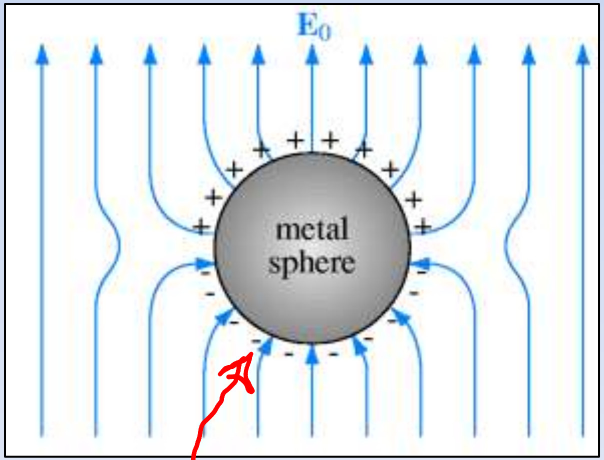


$$\boxed{E_{1t} = 0 = D_{1t}}$$

$$\boxed{D_{1n} = \rho_s = \epsilon_1 E_{1n}}$$

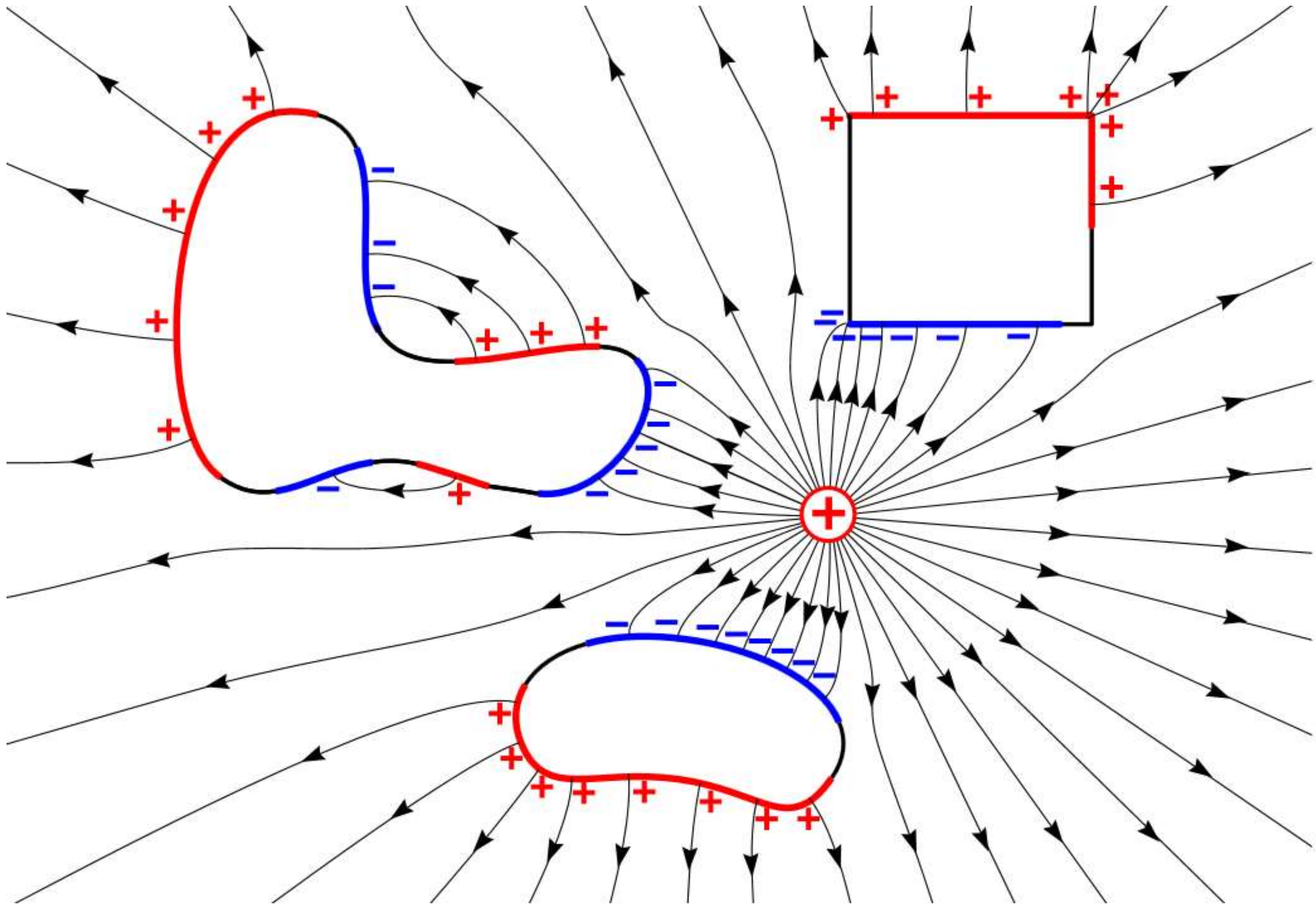
$$\Rightarrow \boxed{\vec{D}_1 = \epsilon_1 \vec{E} = \hat{n}\rho_s}$$

at conductor surface



induced charge

http://en.wikipedia.org/wiki/Electrostatics



The electrostatic field *(lines with arrows)* of a nearby positive charge *(+)* causes the mobile charges in conductive objects to separate due to electrostatic induction. Negative charges *(blue)* are attracted and move to the surface of the object facing the external charge. Positive charges *(red)* are repelled and move to the surface facing away. These induced surface charges are exactly the right size and shape so their opposing electric field cancels the electric field of the external charge throughout the interior of the metal. Therefore the electrostatic field everywhere inside a conductive object is zero, and the electrostatic potential is constant.

Boundary Conditions (yet *more*)

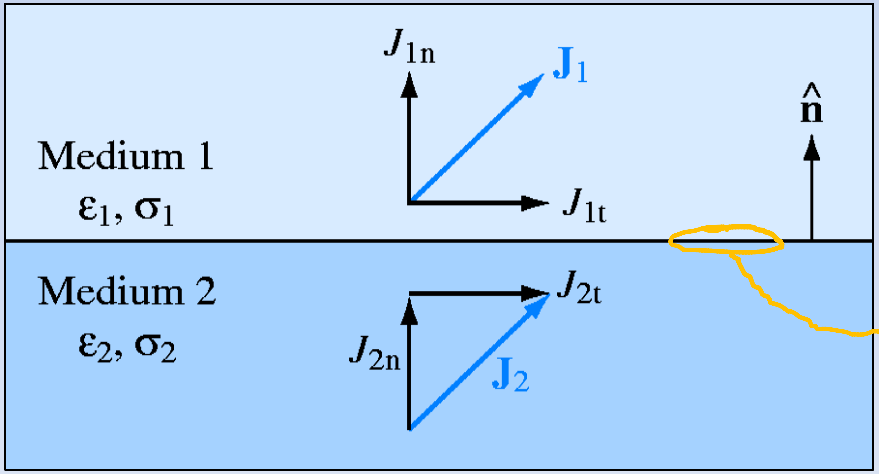
Note: Remember These! ☺

Field Component	Any Two Media	Medium 1 Dielectric ε_1	Medium 2 Dielectric ε_2	Medium 1 Dielectric ε_1	Medium 2 Conductor
Tangential **E**	$E_{1t} = E_{2t}$	$E_{1t} = E_{2t}$		$E_{1t} = E_{2t} = 0$	
Tangential **D**	$D_{1t}/\varepsilon_1 = D_{2t}/\varepsilon_2$	$D_{1t}/\varepsilon_1 = D_{2t}/\varepsilon_2$		$D_{1t} = D_{2t} = 0$	
Normal **E**	$\hat{\mathbf{n}} \cdot (\varepsilon_1 \mathbf{E}_1 - \varepsilon_2 \mathbf{E}_2) = \rho_s$	$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$		$E_{1n} = \rho_s/\varepsilon_1$	$E_{2n} = 0$
Normal **D**	$\hat{\mathbf{n}} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$	$D_{1n} - D_{2n} = \rho_s$		$D_{1n} = \rho_s$	$D_{2n} = 0$

Notes: (1) ρ_s is the surface charge density at the boundary; (2) normal components of $\mathbf{E}_1$, $\mathbf{D}_1$, $\mathbf{E}_2$, and $\mathbf{D}_2$ are along $\hat{\mathbf{n}}_2$, the outward normal unit vector of medium 2.

General Boundary

These still hold ↑ plus $\boxed{J_{1n} = J_{2n}}$

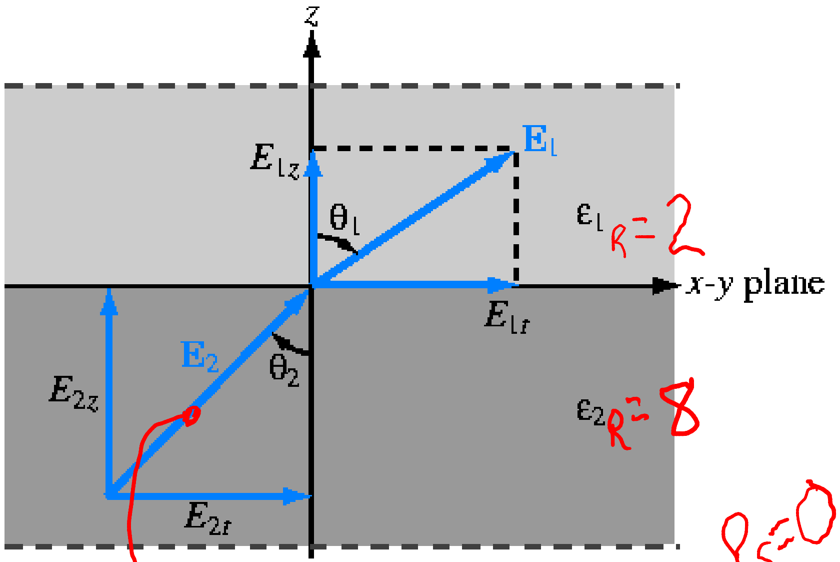


For electrostatics, where ρ_s constant

Recall, $\vec{J} = \sigma \vec{E}$

Neither perfect dielectrics or conductors

(Each actually a bit of both.)



find $\mathbf{E}_1$ if $\mathbf{E}_2 = \hat{\mathbf{x}}2 - \hat{\mathbf{y}}3 + \hat{\mathbf{z}}3$ (V/m), $\varepsilon_1 = 2\varepsilon_0$, and $\varepsilon_2 = 8\varepsilon_0$. Assume the boundary to be charge free.

Given that the x–y plane is the boundary between the two media, the x- and y-components of $\mathbf{E}_2$ are parallel to the boundary, and therefore are the same across the two sides of the boundary. Thus,

$$E_{1x} = E_{2x} = 2$$
$$E_{1y} = E_{2y} = -3.$$

For the z-component,

which is normal to interface

$$\varepsilon_1 E_{1z} = \varepsilon_2 E_{2z} \qquad (\rho_s = 0)$$
$$E_{1z} = \frac{\varepsilon_2}{\varepsilon_1} E_{2z} = \frac{8\varepsilon_0}{2\varepsilon_0} \cdot 3 = 12.$$

Hence,

$$\mathbf{E}_1 = \hat{\mathbf{x}}\,2 - \hat{\mathbf{y}}\,3 + \hat{\mathbf{z}}\,12 \quad \text{(V/m)}.$$

Repeat for a boundary with surface charge density $\rho_s = 3.54 \times 10^{-11}$ (C/m^2).

same as before

$$E_{1x} = 2$$
$$E_{1y} = -3.$$

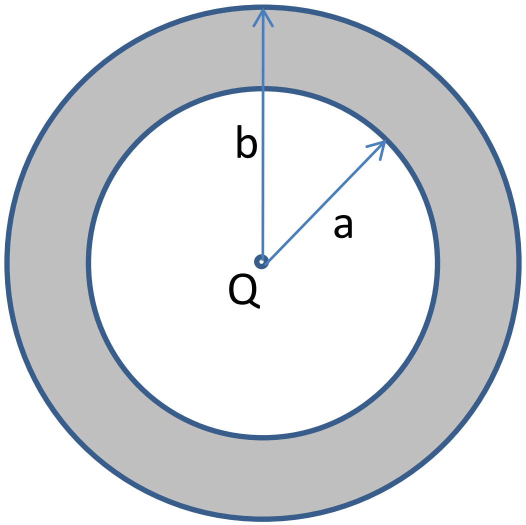
For z-component,

$$\varepsilon_1 E_{1z} - \varepsilon_2 E_{2z} = \rho_s$$
$$E_{1z} = \frac{\varepsilon_2 E_{2z} + \rho_s}{\varepsilon_1}$$
$$= \frac{8\varepsilon_0 \times 3 + 3.54 \times 10^{-11}}{2\varepsilon_0}$$
$$= 12 + \frac{3.54 \times 10^{-11}}{2 \times 8.85 \times 10^{-12}}$$
$$= 12 + 2 = 14 \quad \text{(V/m)}.$$

Hence,

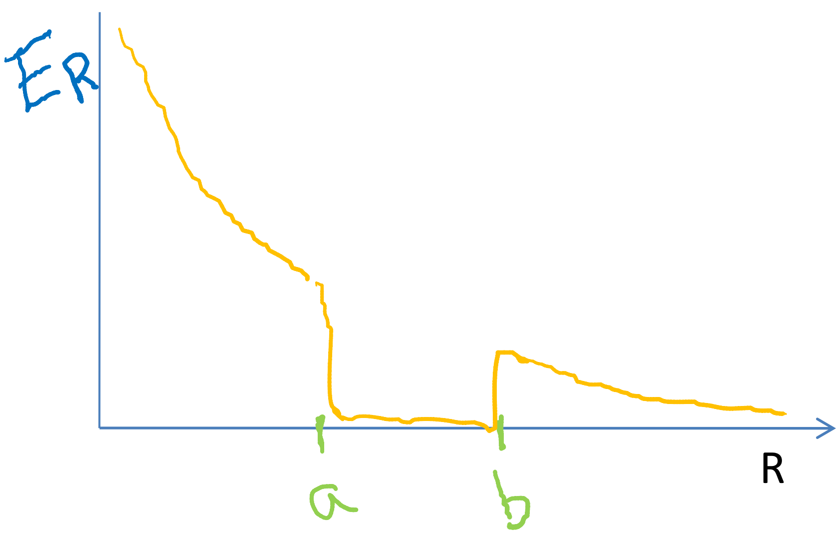
$$\mathbf{E}_1 = \hat{\mathbf{x}}\,2 - \hat{\mathbf{y}}\,3 + \hat{\mathbf{z}}\,14 \quad \text{(V/m)}.$$

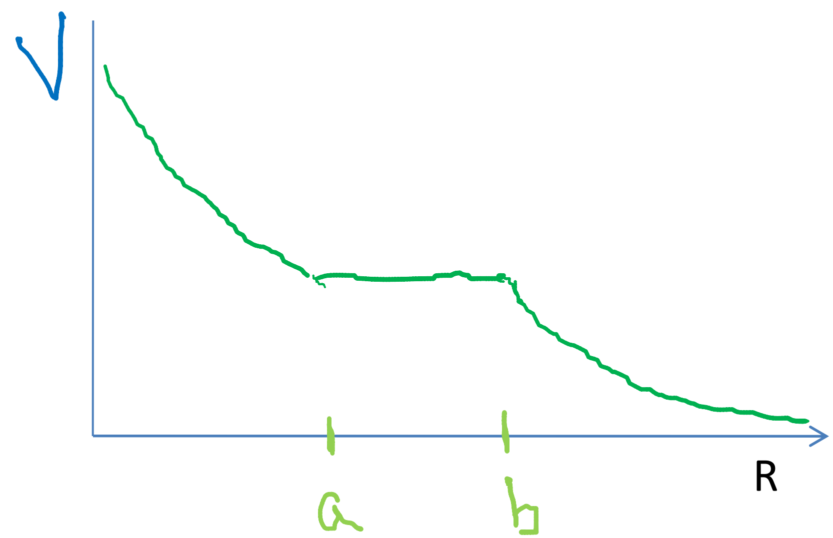
Perfectly conducting spherical shell
(air, inside and out)



Centered at the origin,
where there is a charge *Q*.

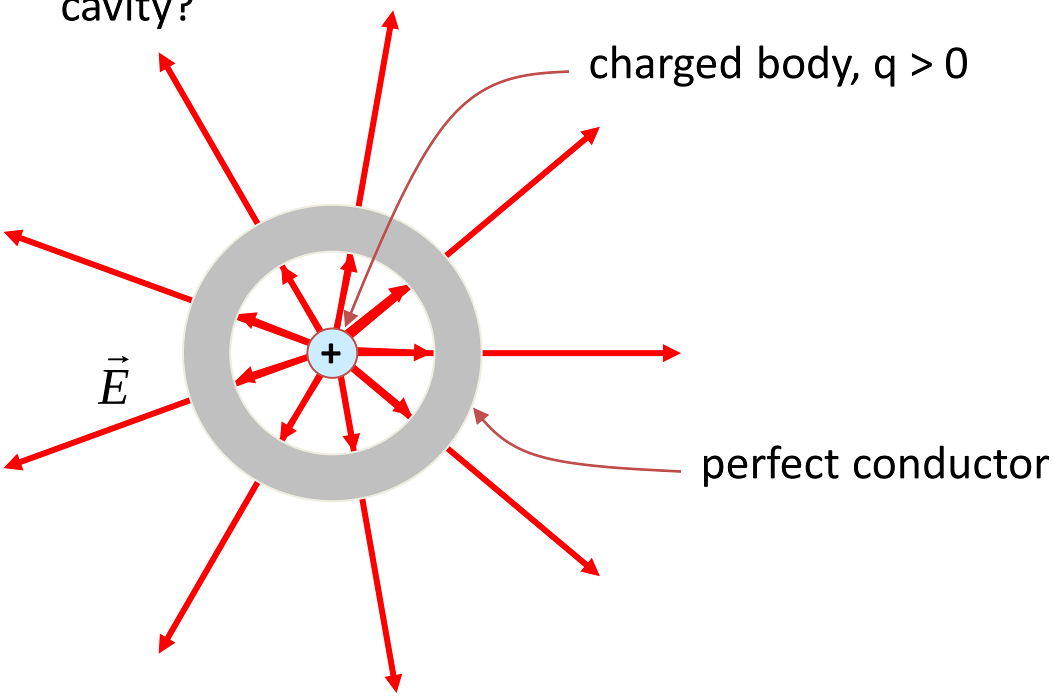
- Describe the Electric field at some point R>b.
- Sketch the radial component of the Electric field vs. R
- Sketch the potential V vs. R

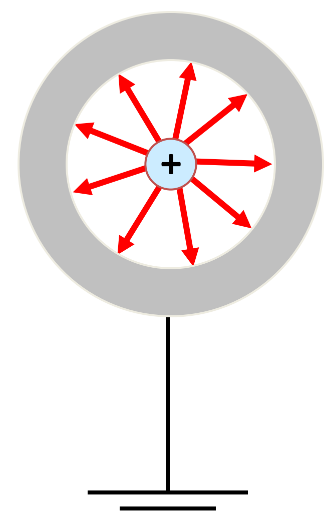




Probing Question

Outside the system shown in the figure, the electric field is as if there was no conducting shell surrounding the charged body. And yet, metal layers are used for shielding static electric fields. How is it possible? What must be done so that the conductor would shield the field of the charged body that is inside the cavity?





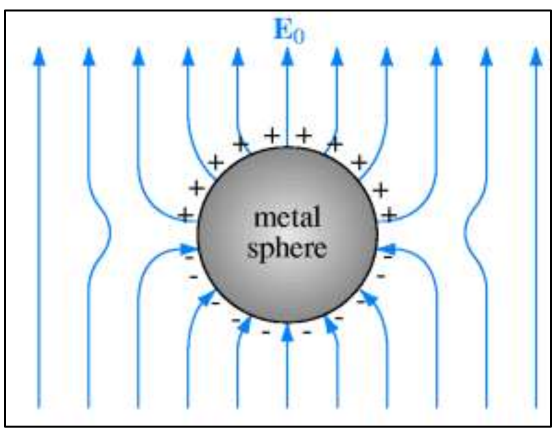
Wrapping up (?) ElectroStatics

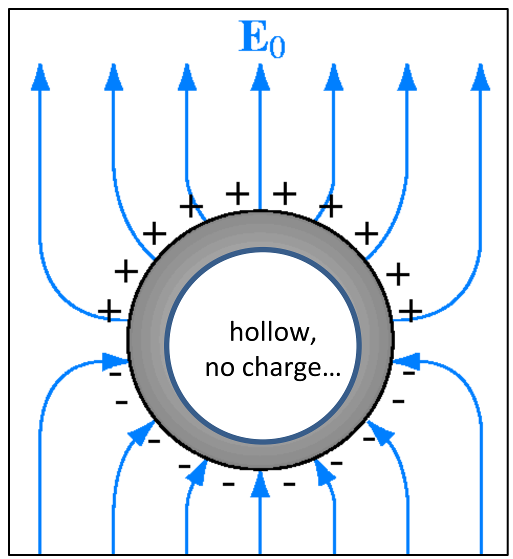
ONLY at interface!!

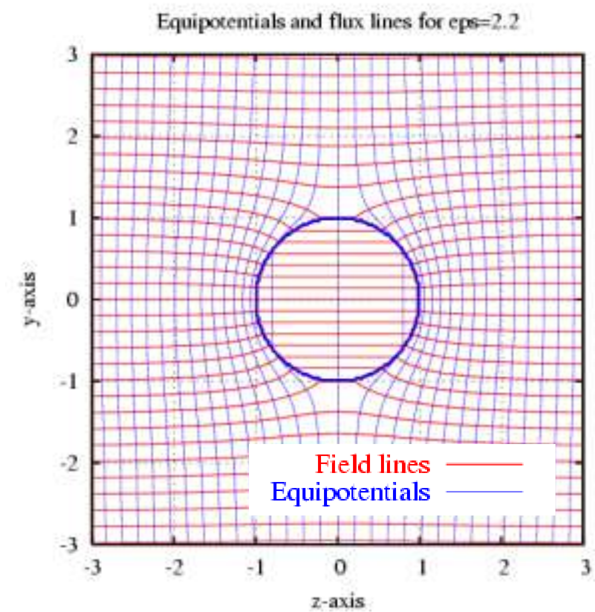
Field Component	Any Two Media	Medium 1 Dielectric ε_1	Medium 2 Dielectric ε_2	Medium 1 Dielectric ε_1	Medium 2 Conductor
Tangential **E**	$E_{1t} = E_{2t}$	$E_{1t} = E_{2t}$		$E_{1t} = E_{2t} = 0$	
Tangential **D**	$D_{1t}/\varepsilon_1 = D_{2t}/\varepsilon_2$	$D_{1t}/\varepsilon_1 = D_{2t}/\varepsilon_2$		$D_{1t} = D_{2t} = 0$	
Normal **E**	$\hat{\mathbf{n}}\cdot(\varepsilon_1\mathbf{E}_1 - \varepsilon_2\mathbf{E}_2) = \rho_s$	$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$		$E_{1n} = \rho_s/\varepsilon_1$	$E_{2n} = 0$
Normal **D**	$\hat{\mathbf{n}}\cdot(\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$	$D_{1n} - D_{2n} = \rho_s$		$D_{1n} = \rho_s$	$D_{2n} = 0$

Notes: (1) ρ_s is the surface charge density at the boundary; (2) normal components of $\mathbf{E}_1$, $\mathbf{D}_1$, $\mathbf{E}_2$, and $\mathbf{D}_2$ are along $\hat{\mathbf{n}}_2$, the outward normal unit vector of medium 2.

What does *potential* do at interface?

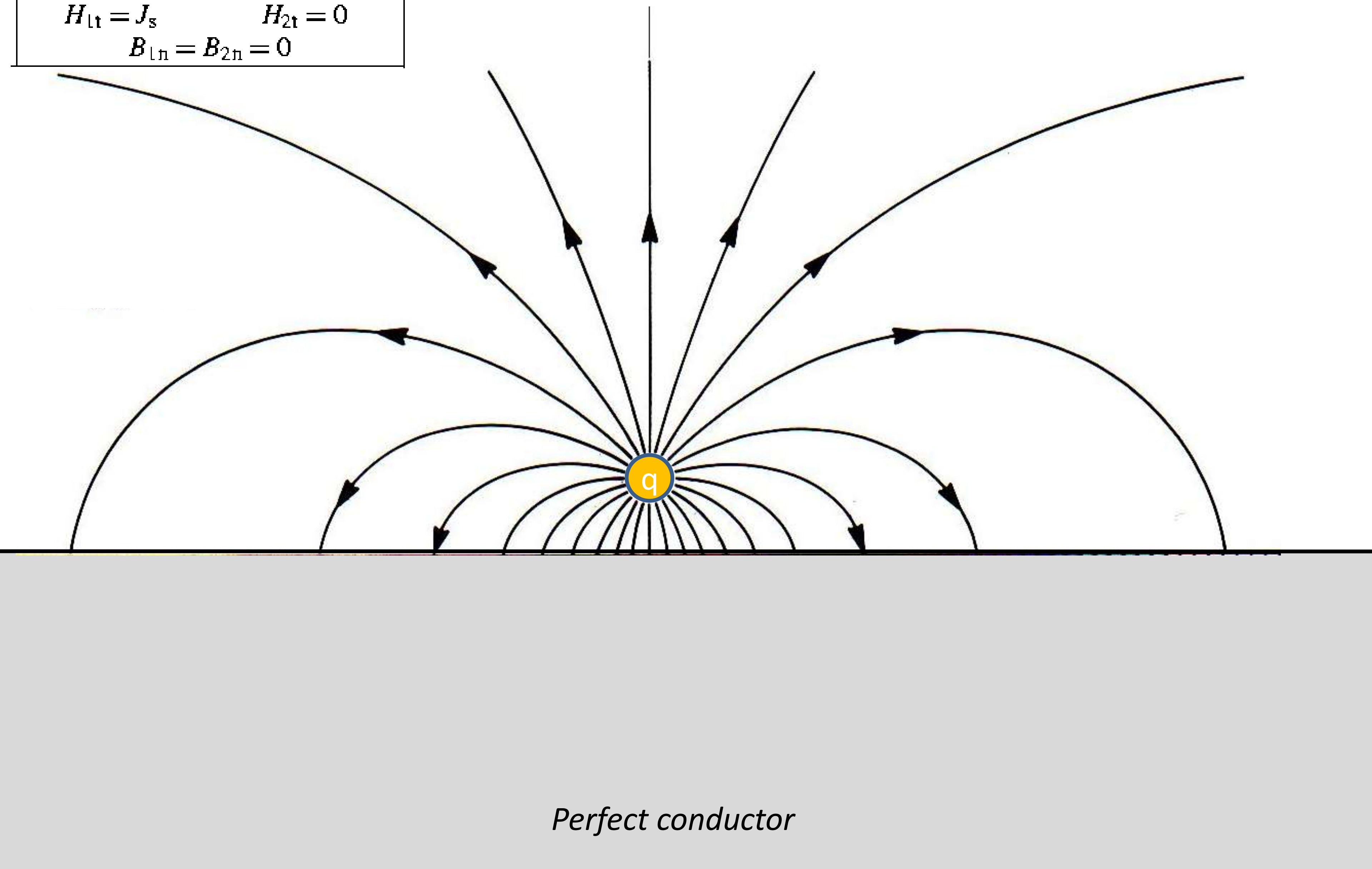






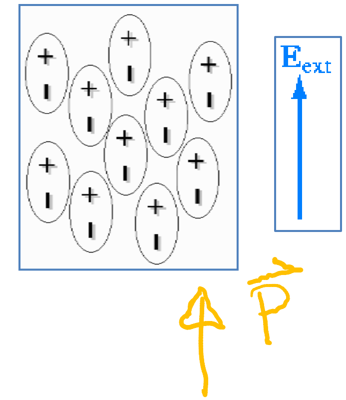
Faraday cage, etc.

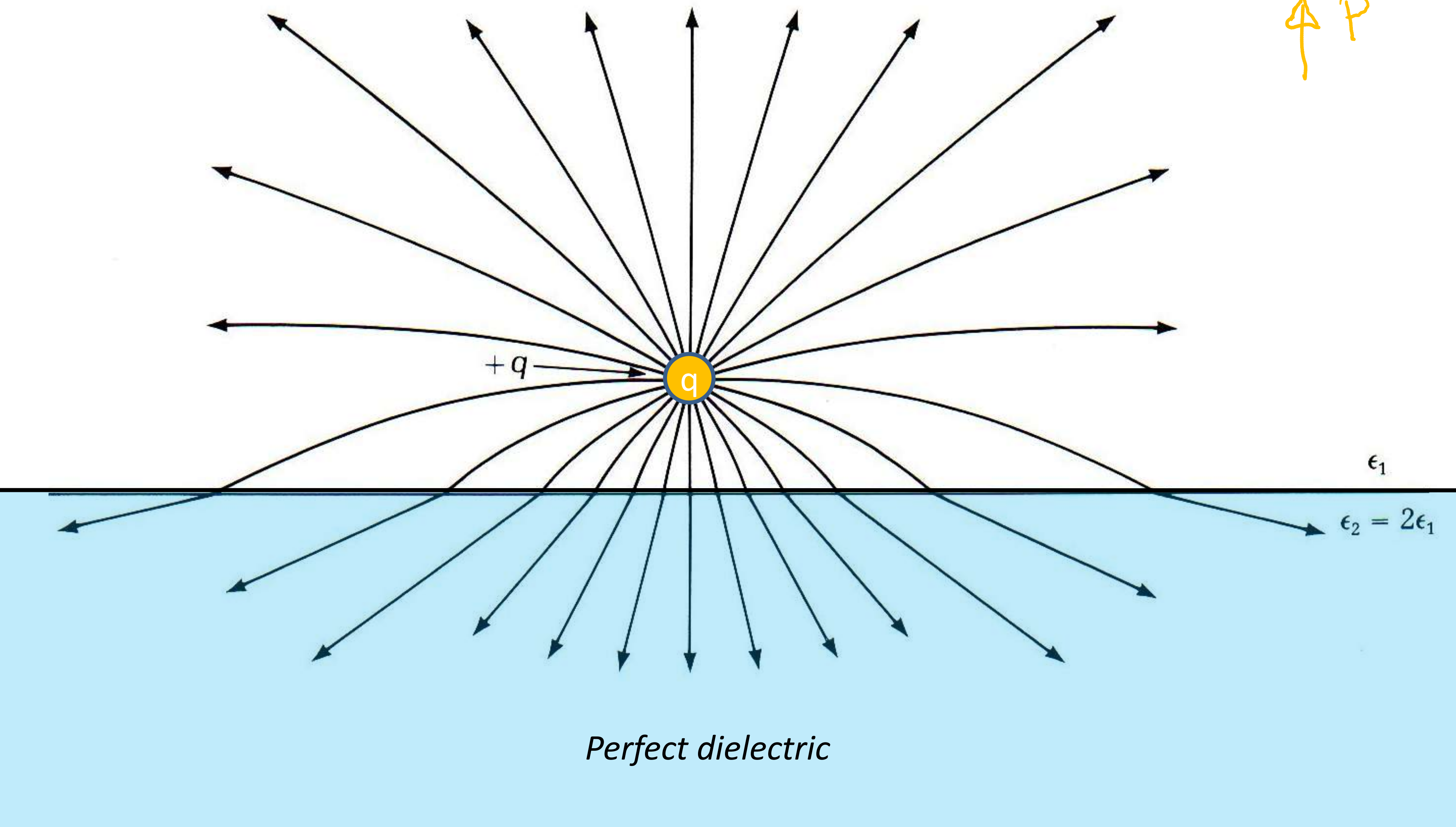
Medium 1 Dielectric	Medium 2 Conductor
$E_{1t} = E_{2t} = 0$	
$D_{1n} = \rho_s$	$D_{2n} = 0$
$H_{1t} = J_s$	$H_{2t} = 0$
$B_{1n} = B_{2n} = 0$	



Medium 1 Dielectric	Medium 2 Dielectric
$E_{1t} = E_{2t}$	
$D_{1n} - D_{2n} = \rho_s$	
$H_{1t} = H_{2t}$	
$B_{1n} = B_{2n}$	

$E_{2n} = \frac{\epsilon_1}{\epsilon_2} E_{1n}$ for $\rho_s = 0$





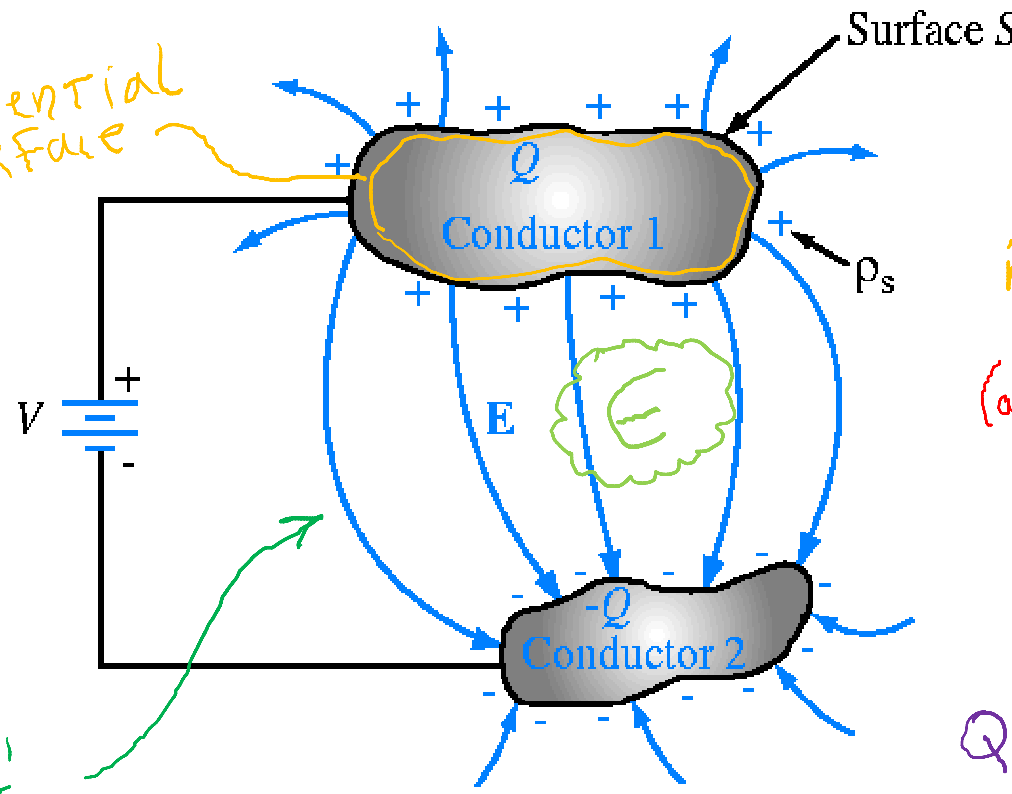
Capacitance

$$C = \frac{Q}{V}$$

$$\boxed{\frac{C}{V} = F}$$

Independent of Geometry ?

Equipotential Surface



$$\hat{n}\cdot\vec{E} = \frac{\rho_s}{\epsilon}$$

(at conductor surface)

$$\Downarrow$$

$$Q = \int_S \rho_s \, ds$$

$$= \int_S \epsilon \vec{E}\cdot d\vec{s}$$

$$V_{12} = -\int_{P_2}^{P_1} \vec{E}\cdot d\vec{l}$$

any path

any points on conductors

Capacitance

LaPLace

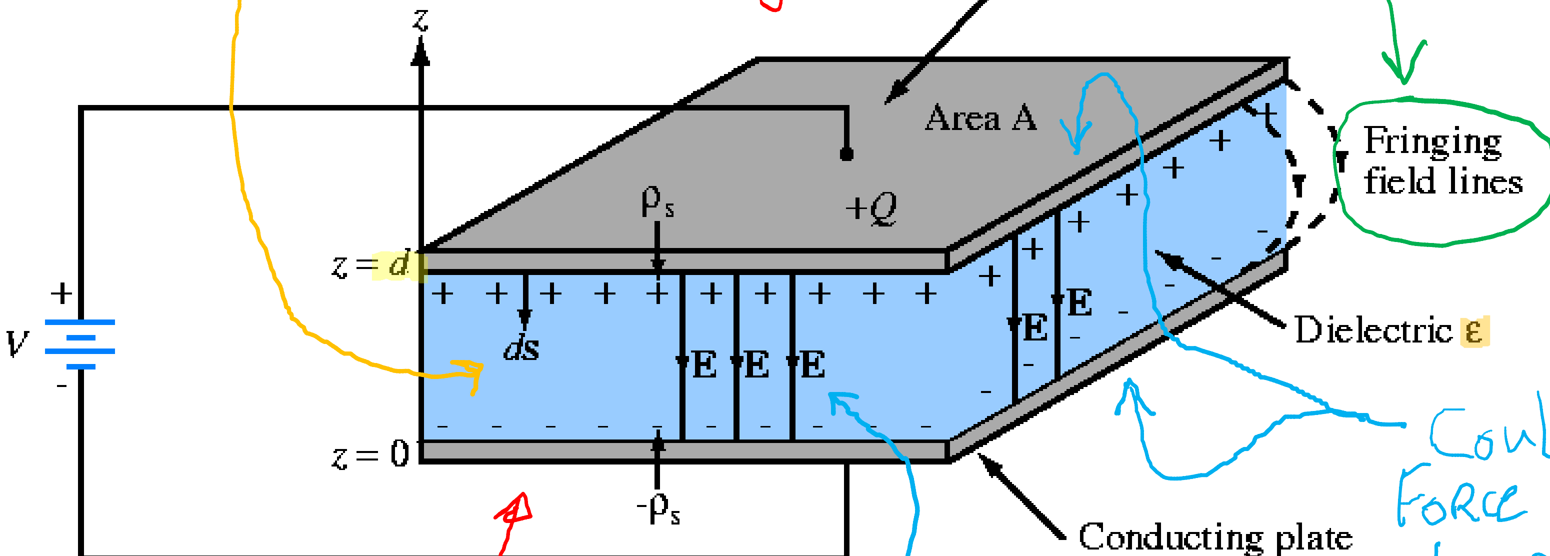
$$\nabla^2 V = 0 \cong \frac{d^2}{dz^2} V$$

$$\Rightarrow V(z) = V_s z + V_0$$

$$= \frac{V}{d} z$$

for given ground

Ignore these if plate dimensions >> d



Coulomb Force wants to pull these together!

$$V = -\int_0^d \vec{E} \cdot d\vec{l} = Ed$$

$$\vec{E} = -\hat{z} E$$

(Energy Storage)

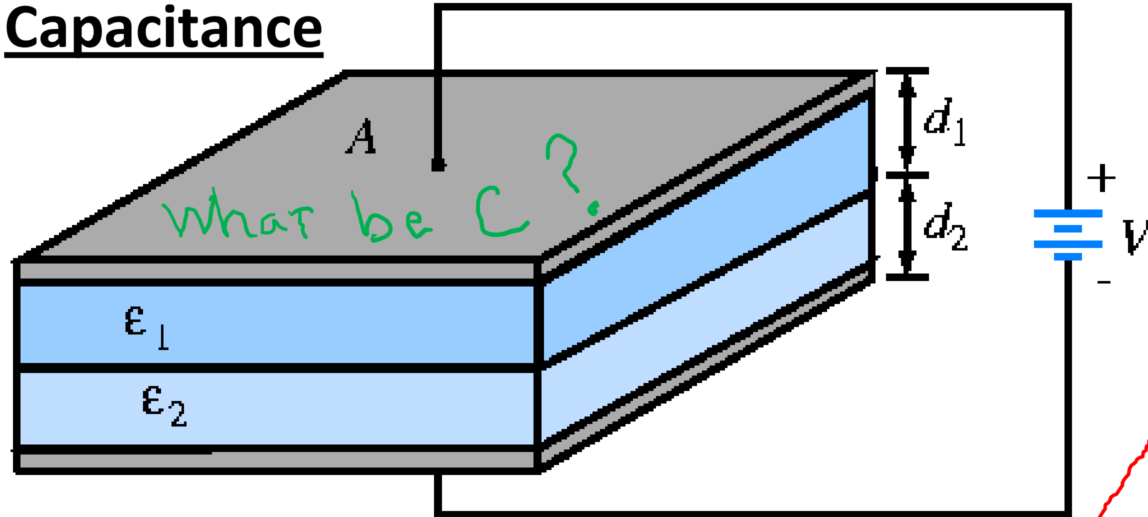
$$E = \frac{\rho_s}{\epsilon} = \frac{Q}{A} \Big/ \epsilon$$

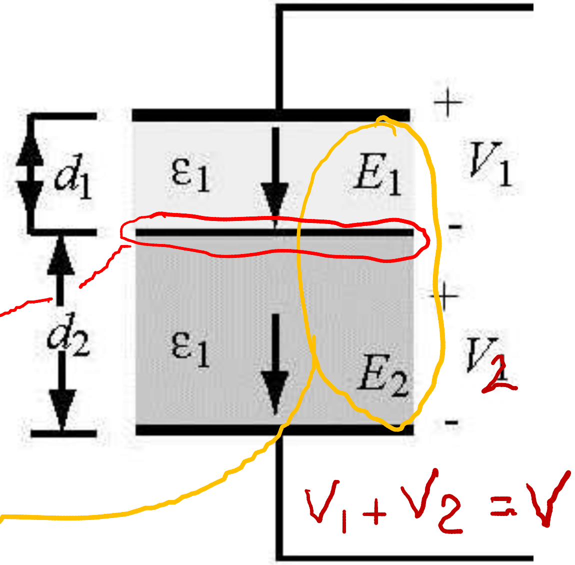
Indy of V

$$\boxed{C = \frac{Q}{V} = \frac{\epsilon A}{d}}$$

Design choices ☺

Capacitance





$V_1 + V_2 = V$

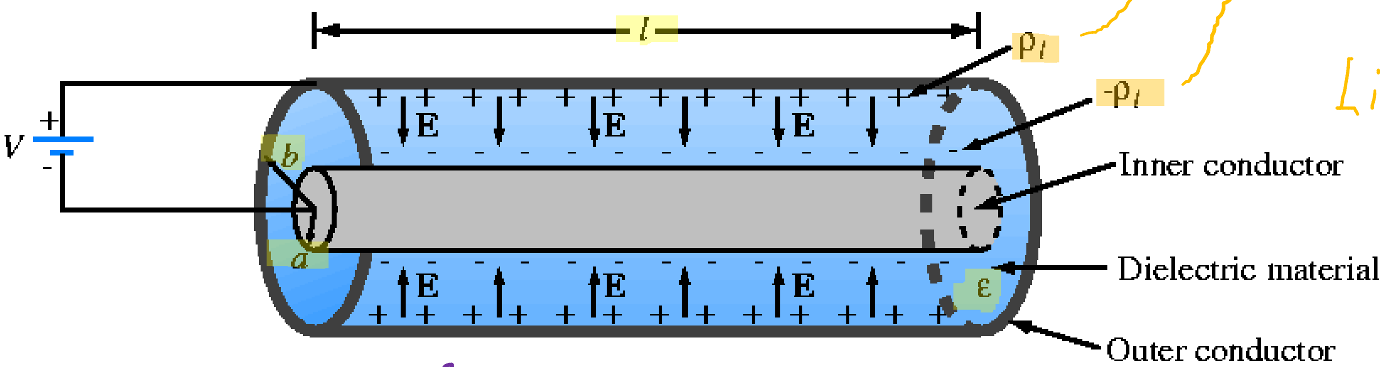
$E_1 = \frac{V_1}{d_1}$, $E_2 = V_2 / d_2$

$\vec{D}$ is continuous across this (charge free) boundary

$\Rightarrow D_{1n} = D_{2n} \Rightarrow \epsilon_1 E_1 = \epsilon_2 E_2 \Rightarrow E_1 = \dfrac{V}{d_1 + \frac{\epsilon_1}{\epsilon_2} d_2}$

but $E_1 = \left(\frac{Q}{A}\right) / \epsilon_1 \Rightarrow$ $C = \dfrac{Q}{V} = \dfrac{A}{\frac{d_1}{\epsilon_1} + \frac{d_2}{\epsilon_2}}$

Capacitance



$Q, -Q$ on inner/outer

OR

Line charge on inner/outer

OR

surface charge density

is different

$$C = \frac{2\pi\epsilon l}{\ln(b/a)} = \frac{Q}{V} = \frac{\epsilon \oint_S \vec{E}\cdot d\vec{s}}{-\int_l \vec{E}\cdot d\vec{l}}$$

$\Rightarrow E_{n\,\text{inner}} > E_{n\,\text{outer}}$

Stored Energy

incremental

$$\frac{1}{2}\epsilon|E|^2$$

Total

$$\frac{1}{2}\int_V \epsilon|E|^2 dv = \frac{1}{2}CV^2$$

Q: What about that whole $i = C\frac{dv}{dt}$ thing from EE 210??

Example 4-12: Capacitance Per Unit Length of Coaxial Line

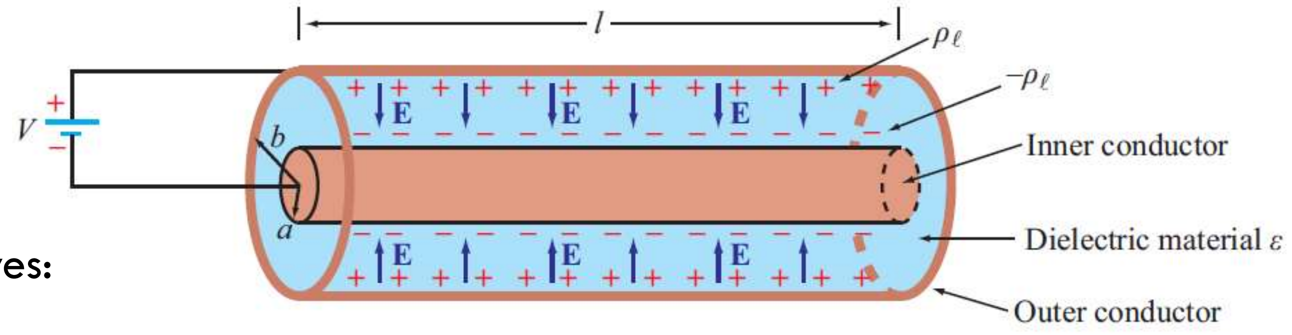


Figure 4-25: Coaxial capacitor filled with insulating material of permittivity ε (Example 4-12).

Application of Gauss's law gives:

$$\mathbf{E} = -\hat{\mathbf{r}} \, \frac{Q}{2\pi\varepsilon r l} \, .$$

The potential difference V between the outer and inner conductors is

$$V = -\int_a^b \mathbf{E} \cdot d\mathbf{l} = -\int_a^b \left(-\hat{\mathbf{r}} \, \frac{Q}{2\pi\varepsilon r l} \right) \cdot (\hat{\mathbf{r}} \, dr)$$

$$= \frac{Q}{2\pi\varepsilon l} \ln\left(\frac{b}{a}\right). \qquad (4.115)$$

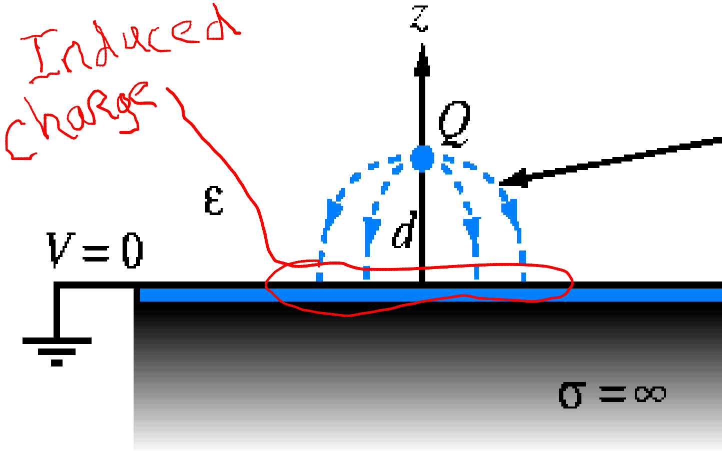
Q is total charge on inside of outer cylinder, and –Q is on outside surface of inner cylinder

The capacitance C is then given by

$$\boxed{C = \frac{Q}{V} = \frac{2\pi\varepsilon l}{\ln(b/a)} \, , \qquad (4.116)}$$

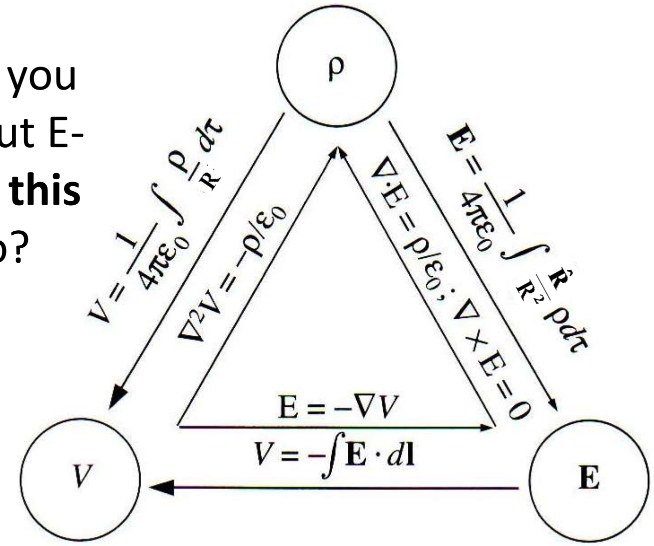
and the capacitance per unit length of the coaxial line is

$$C' = \frac{C}{l} = \frac{2\pi\varepsilon}{\ln(b/a)} \quad \text{(F/m)}. \qquad (4.117)$$

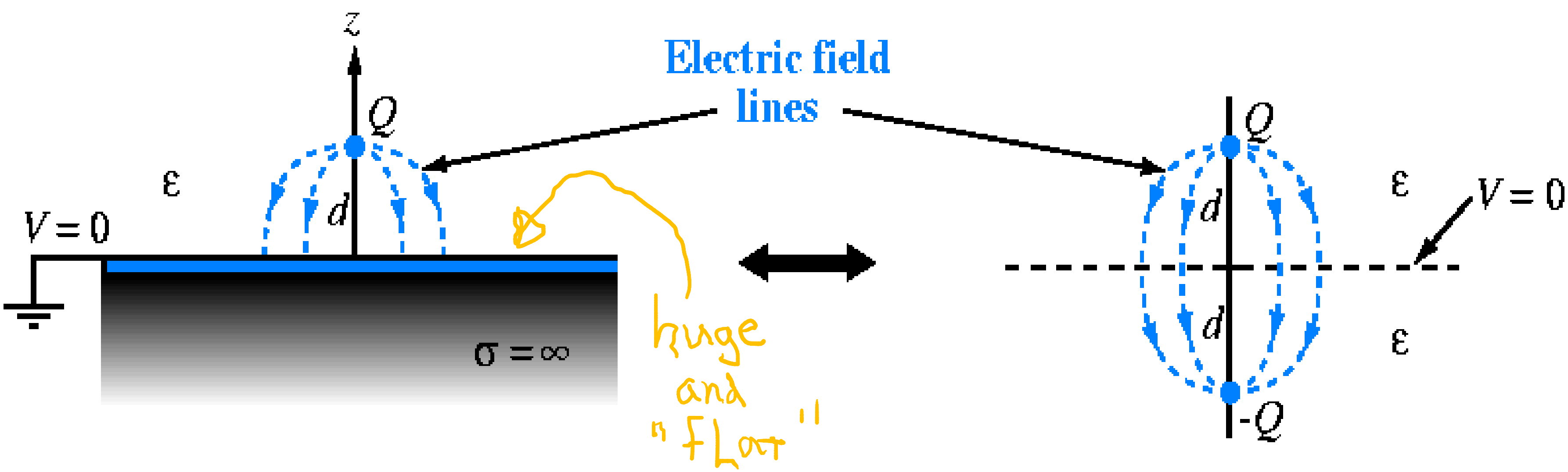


Charge Q above grounded plane

How do you figure out E-field for **this** scenario?

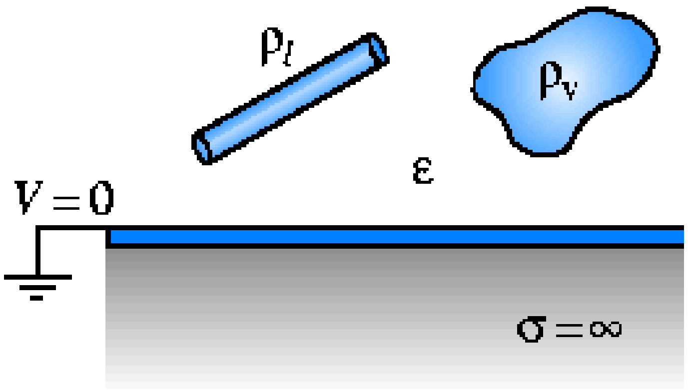


Image

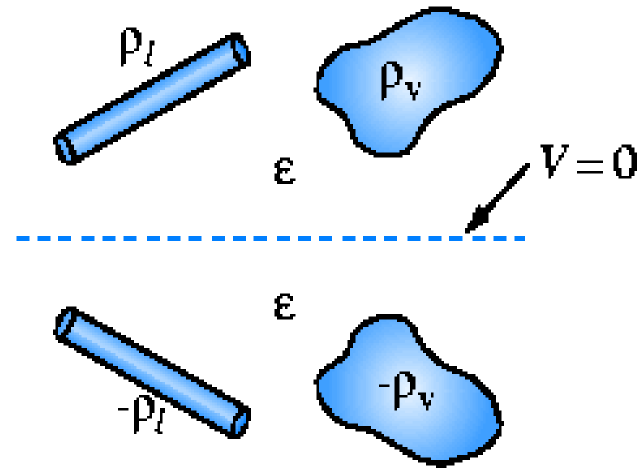


Charge Q above grounded plane

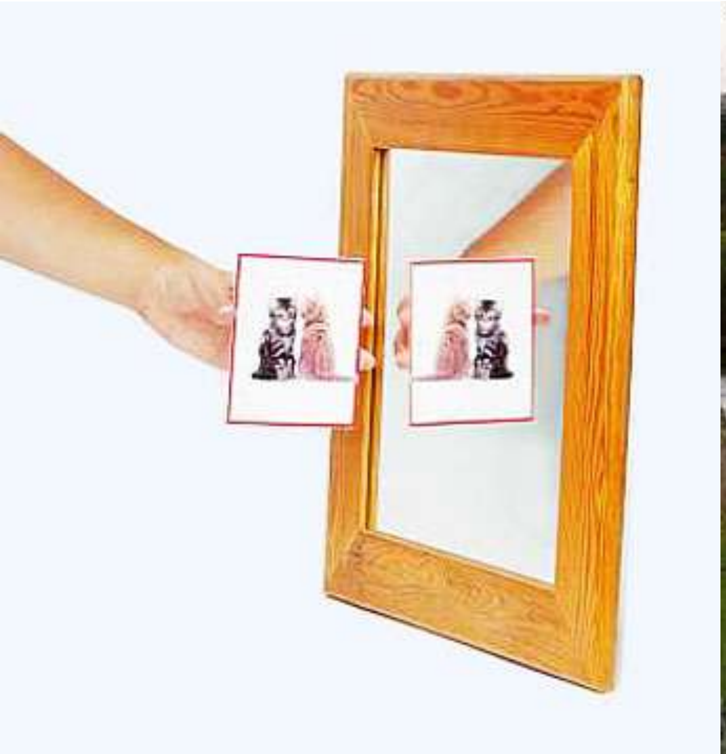
Equivalent configuration

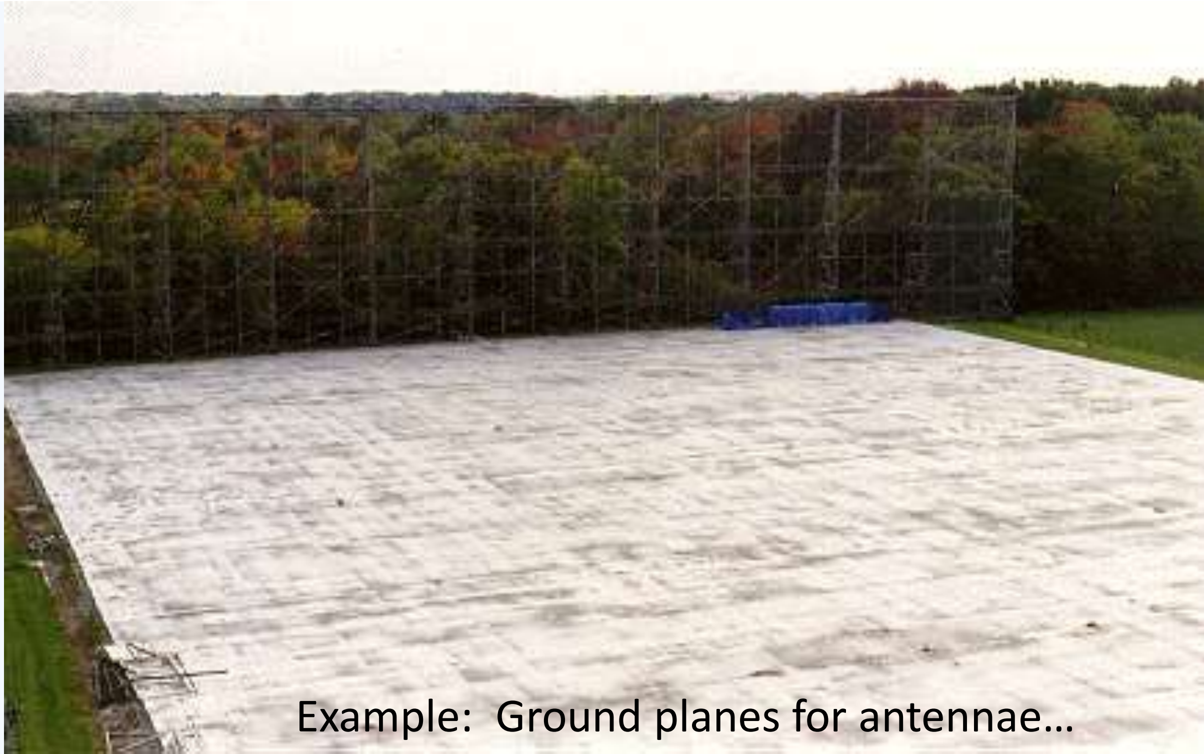


Charge distributions above ground plane



Equivalent distributions





Example: Ground planes for antennae…